Interactive Formal Verification
10: Structured Proofs

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Computer Laboratory
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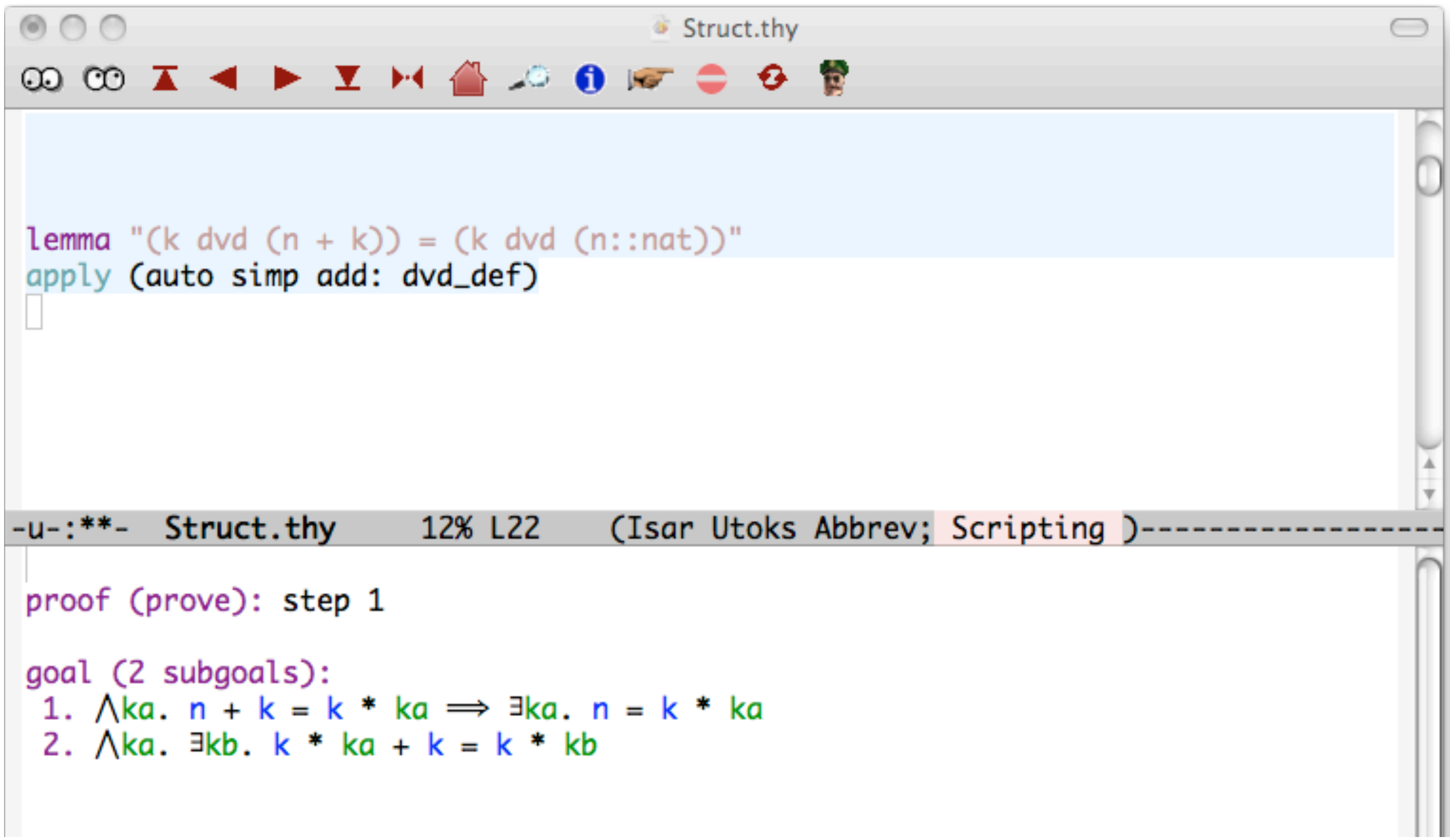
A Proof about "Divides"

$$b \text{ dvd } a \leftrightarrow (\exists k.\ a = b \times k)$$



A Proof about "Divides"

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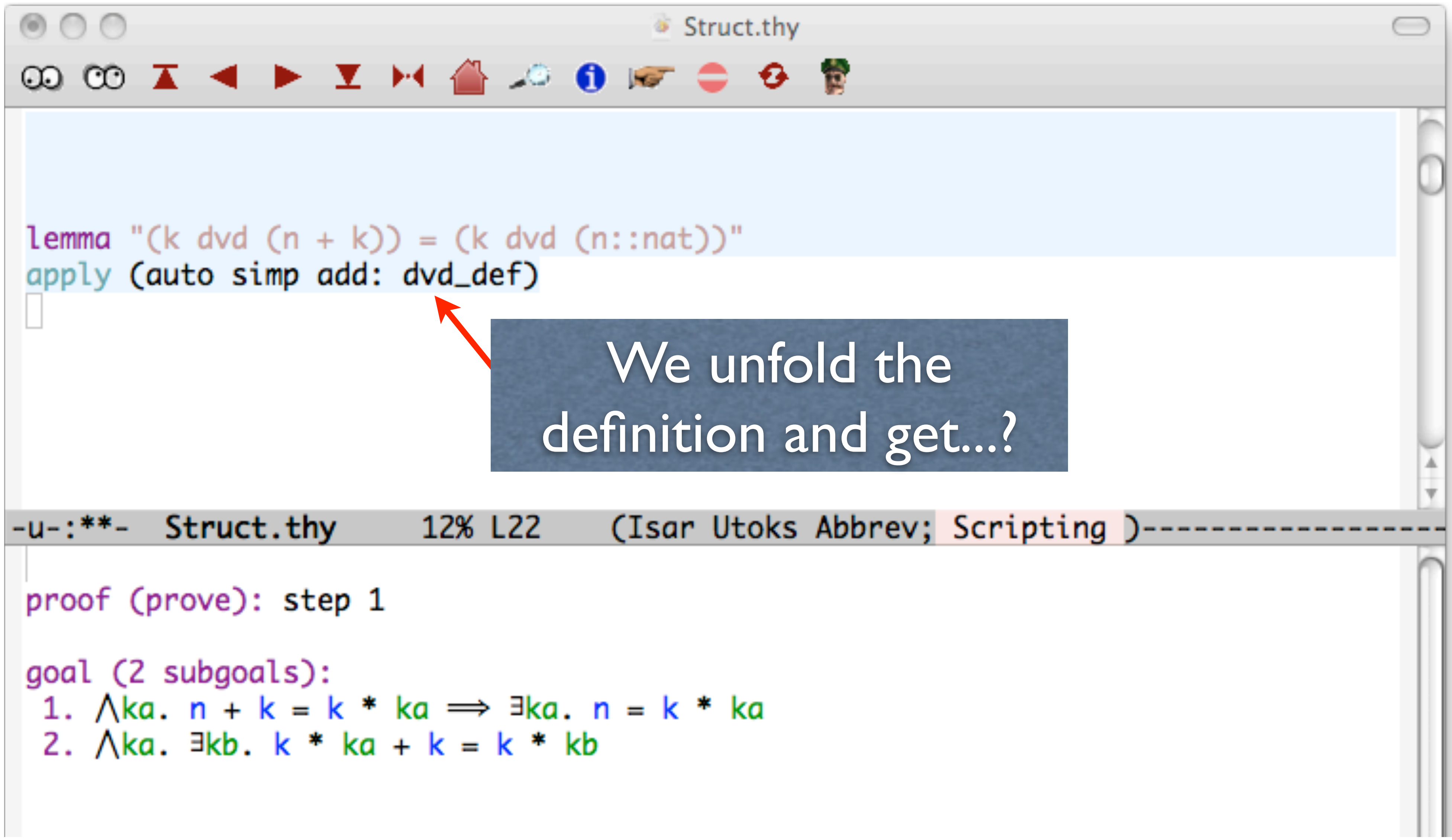

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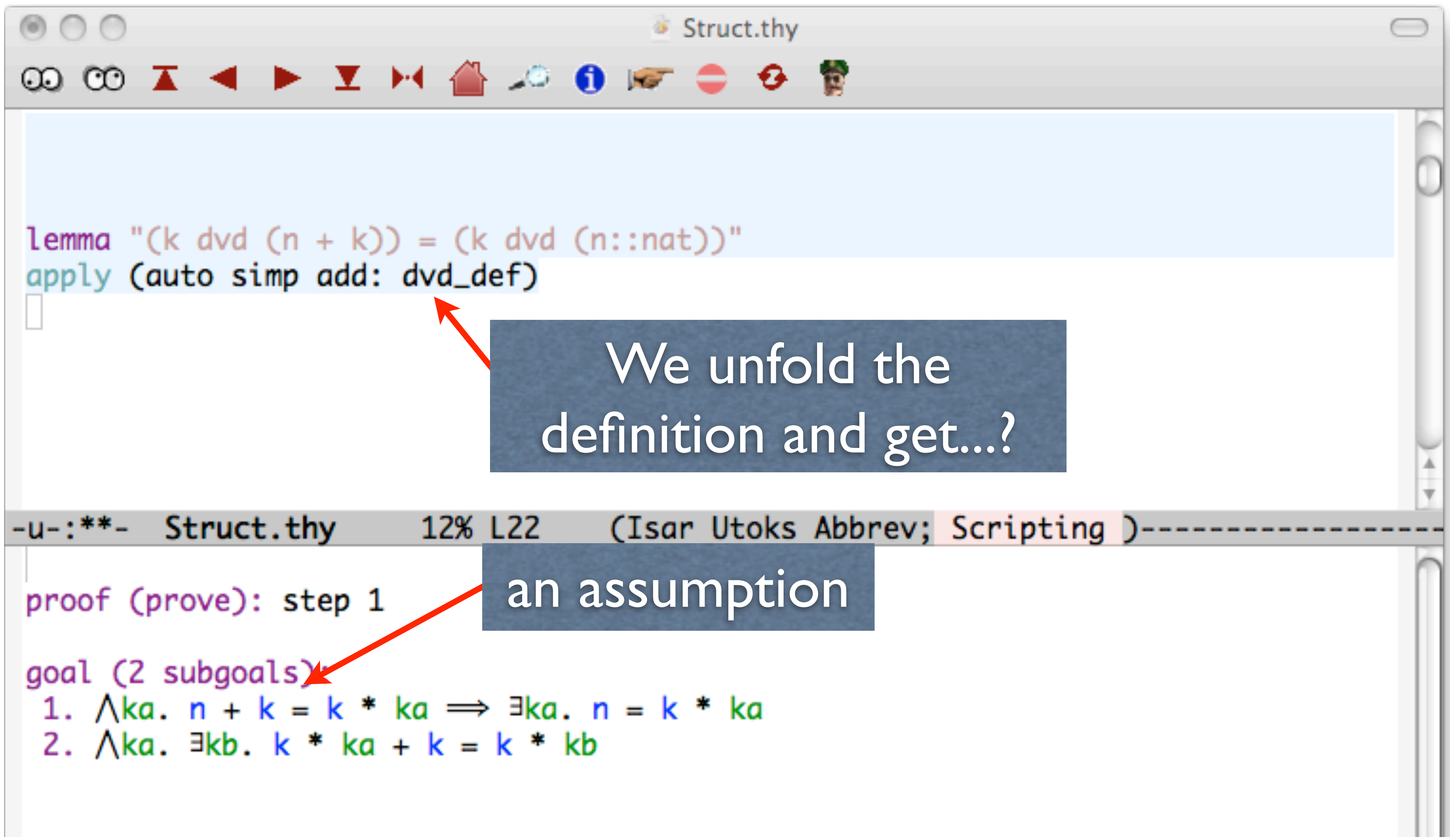

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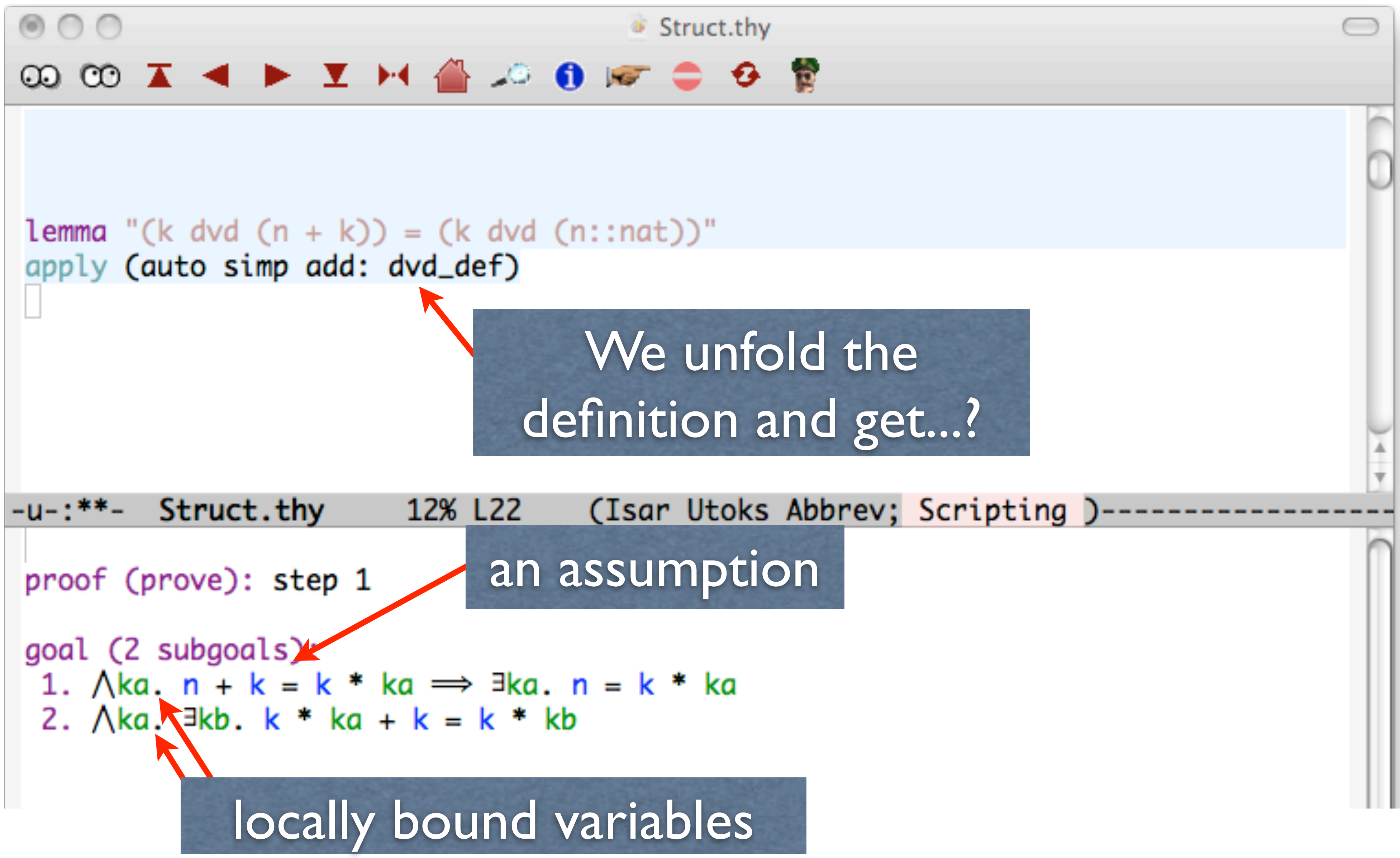

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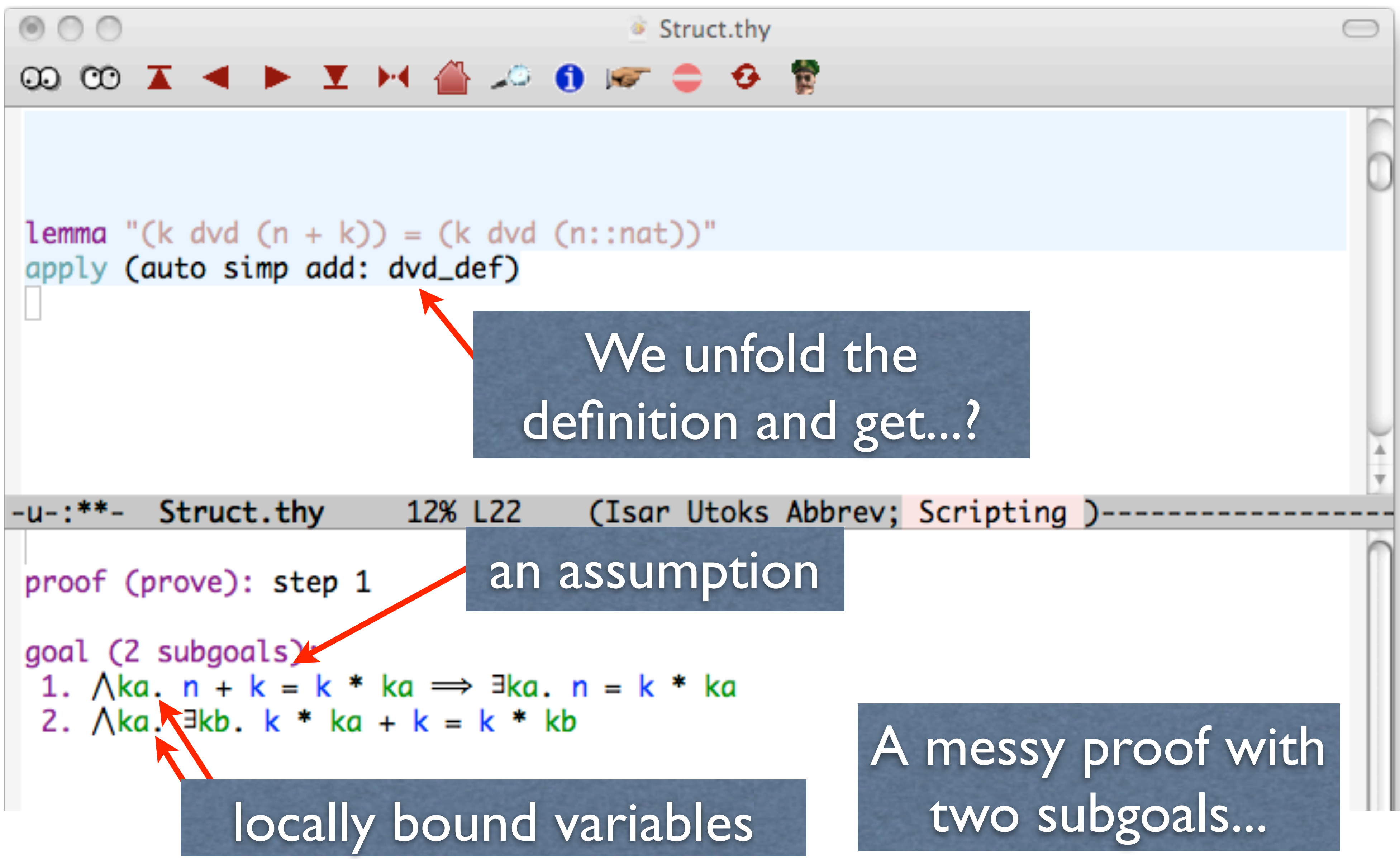

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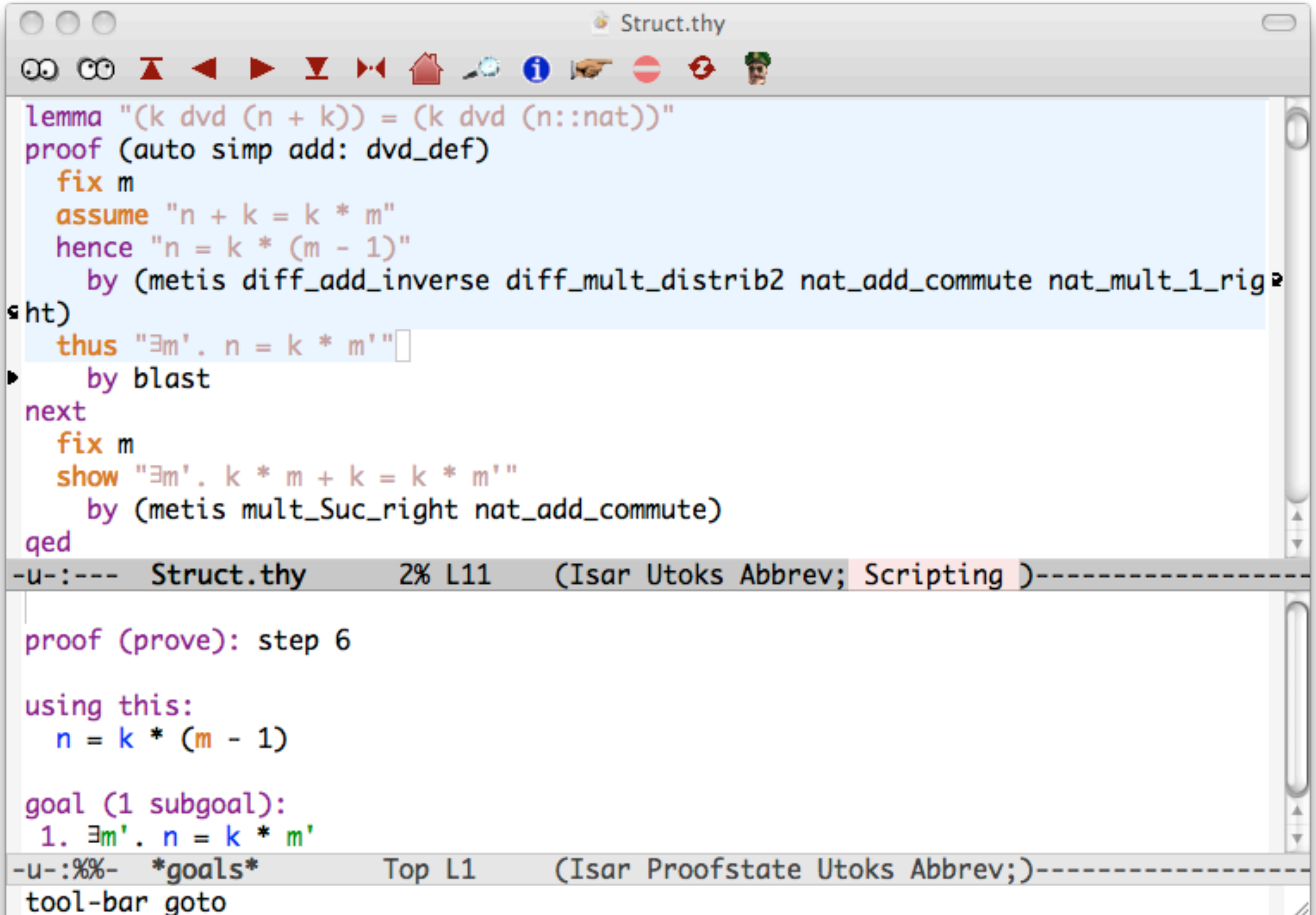
- Isabelle provides many tactics that refer to bound variables and assumptions.
  - Assumptions are often found by matching.
  - Bound variables can be referred to by name, but these names are fragile.
- *Structured proofs* provide a robust means of referring to these elements by name.
- Structured proofs are typically verbose but much more readable than linear apply-proofs.

A Structured Proof



But how do you write them?

The Elements of Isar

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- A *proof context* holds local variables and assumptions of a subgoal.
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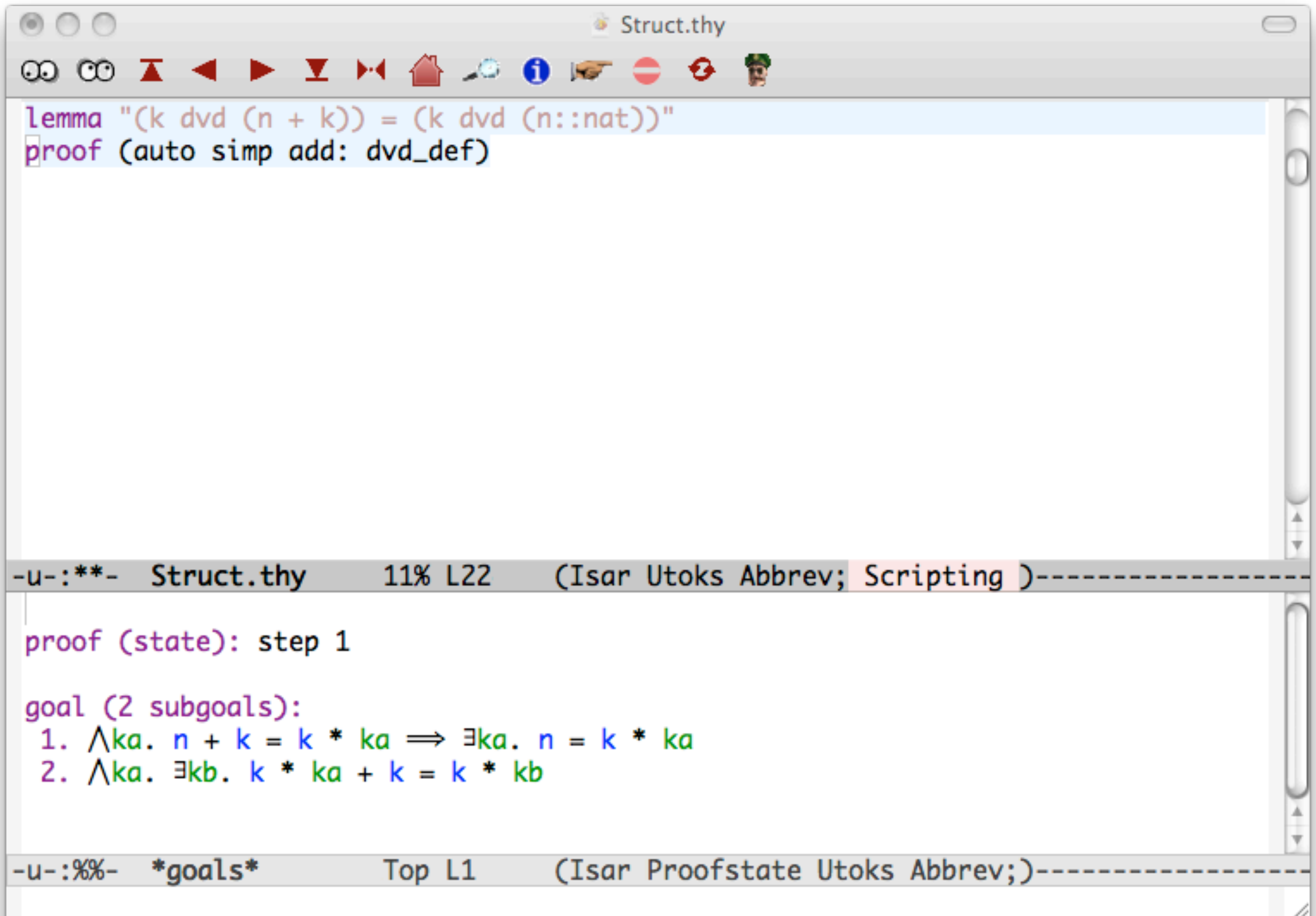
    - In a context, the variables are free and the assumptions are simply theorems.
    - Closing a context yields a theorem having the structure of a subgoal.
- The Isar language lets us state and prove intermediate results, express inductions, etc.

Getting Started



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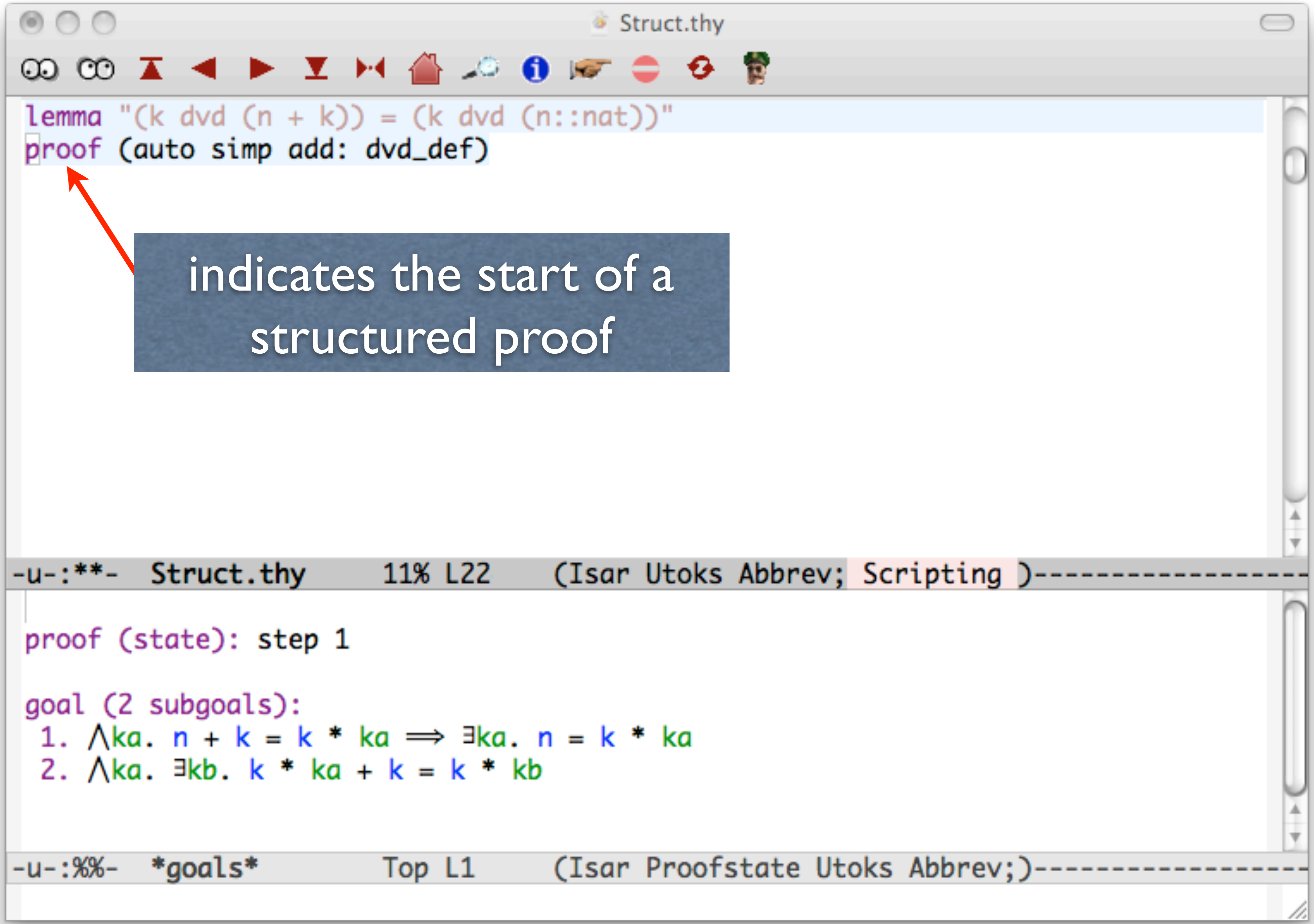
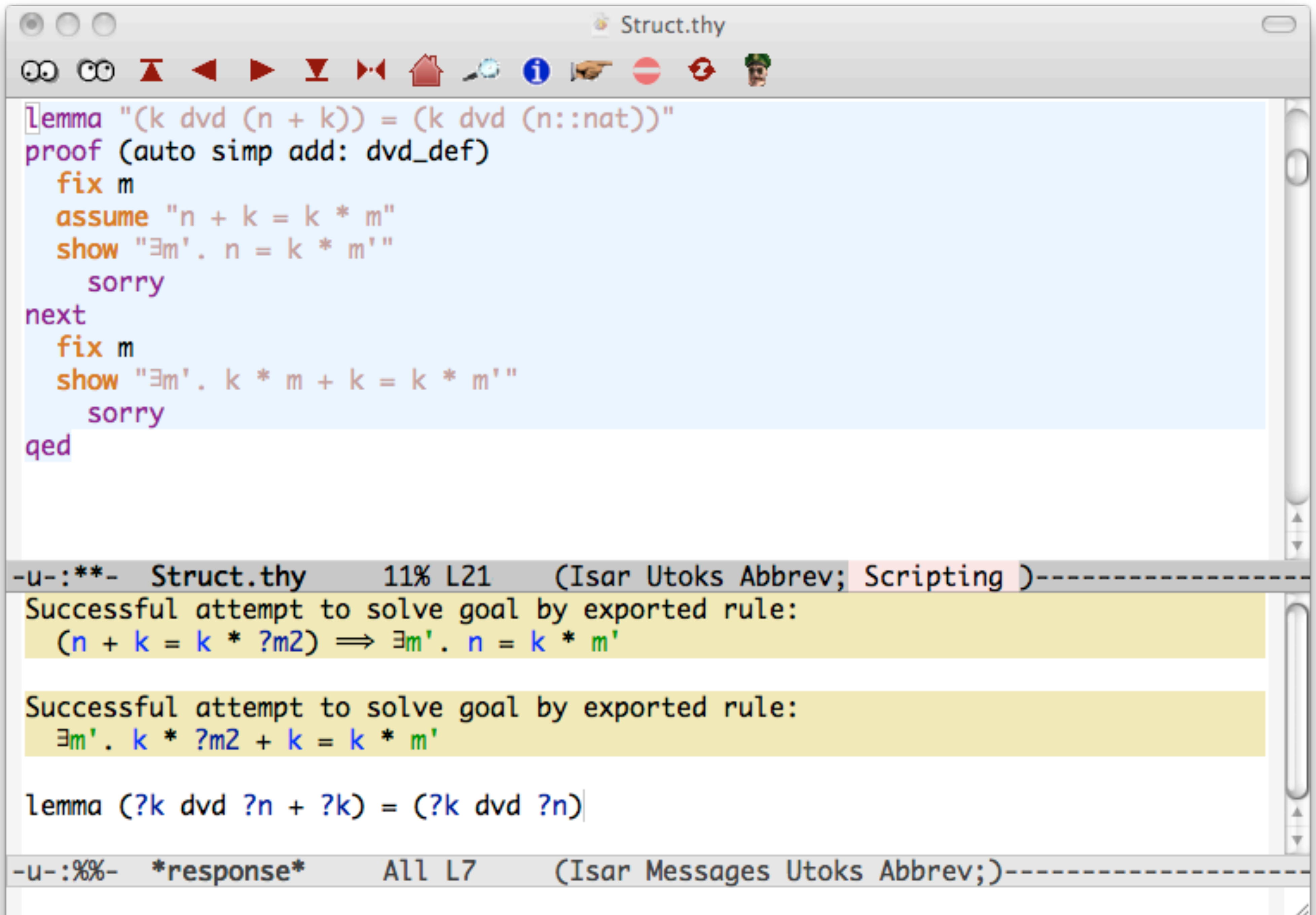


The Proof Skeleton



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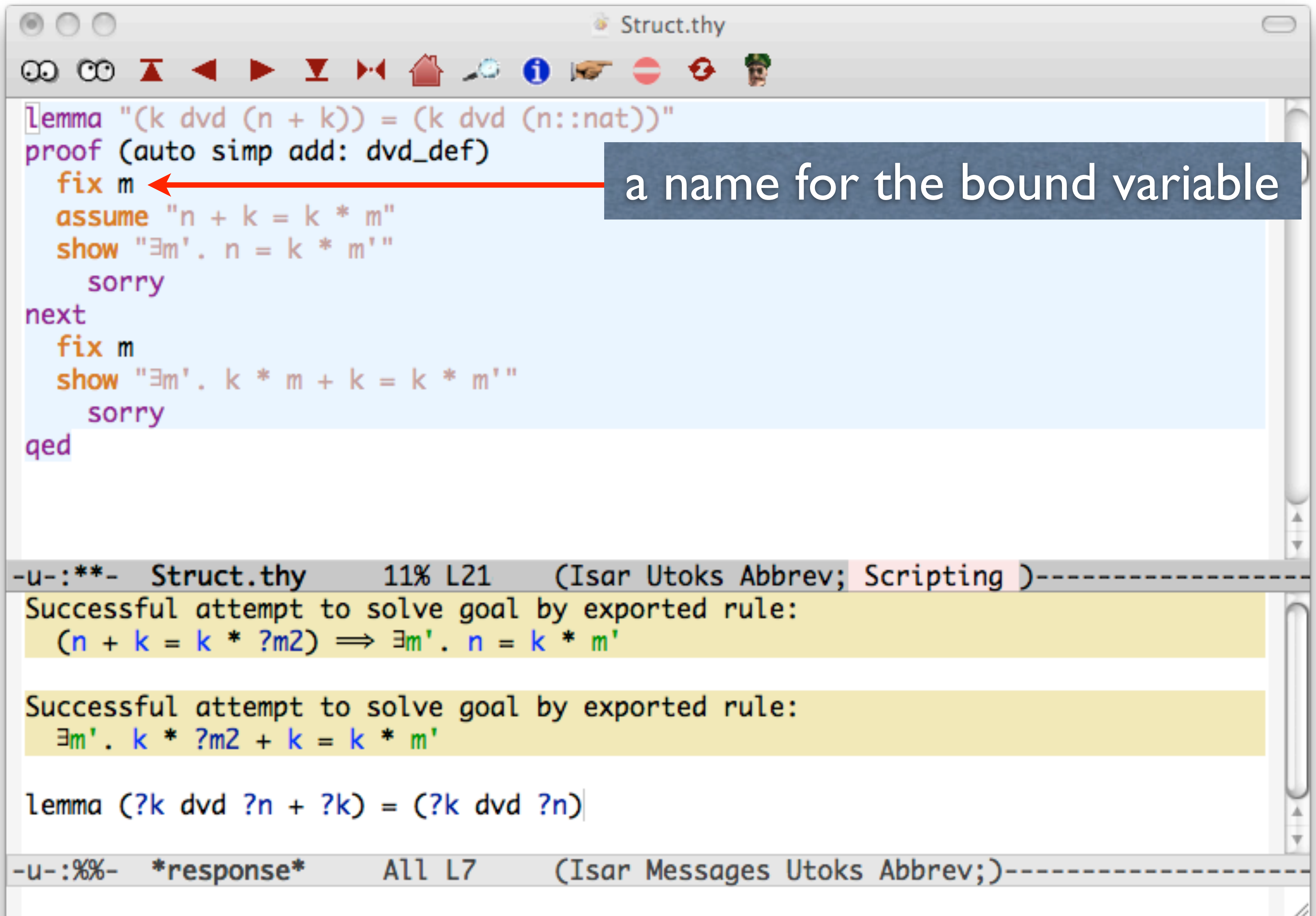

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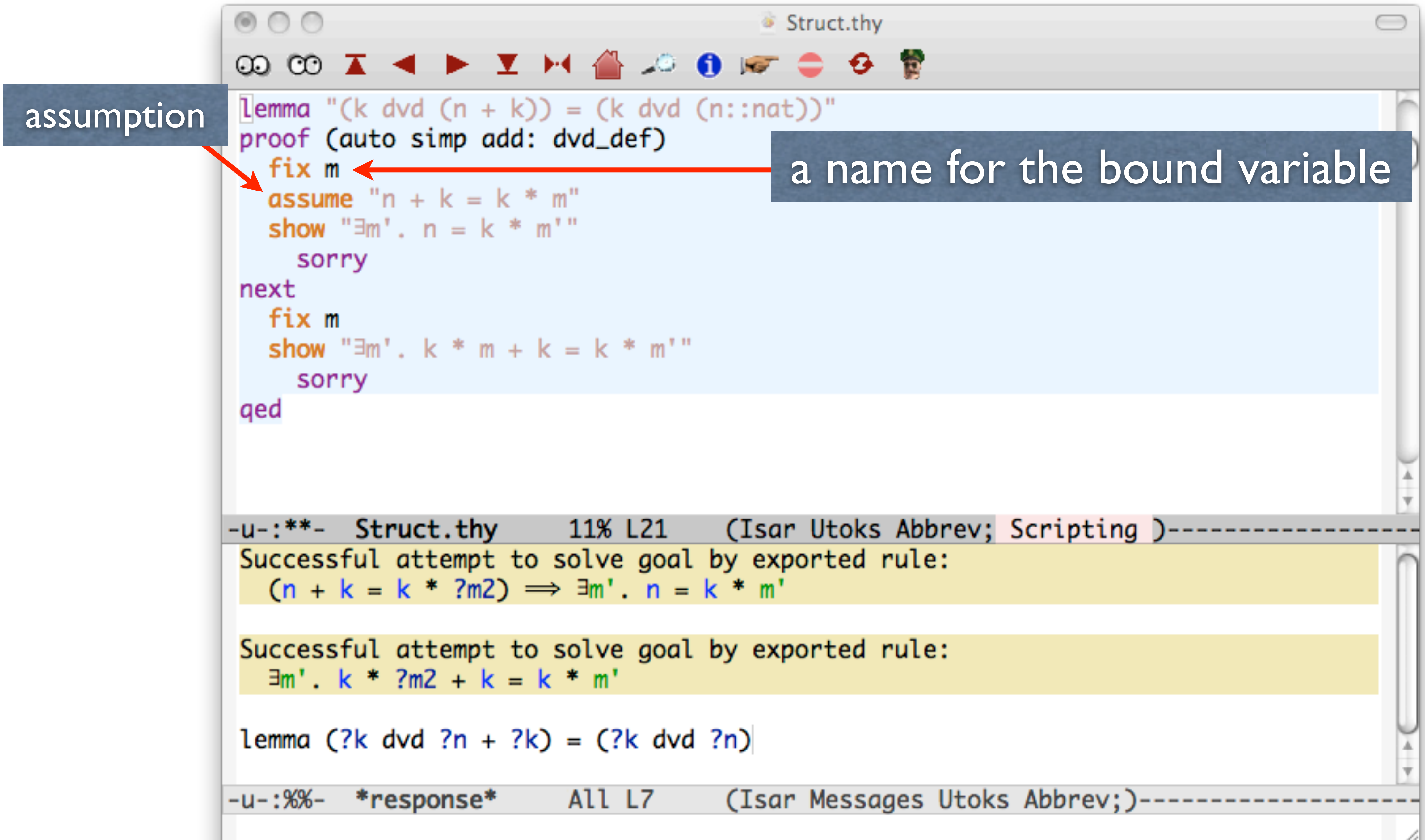

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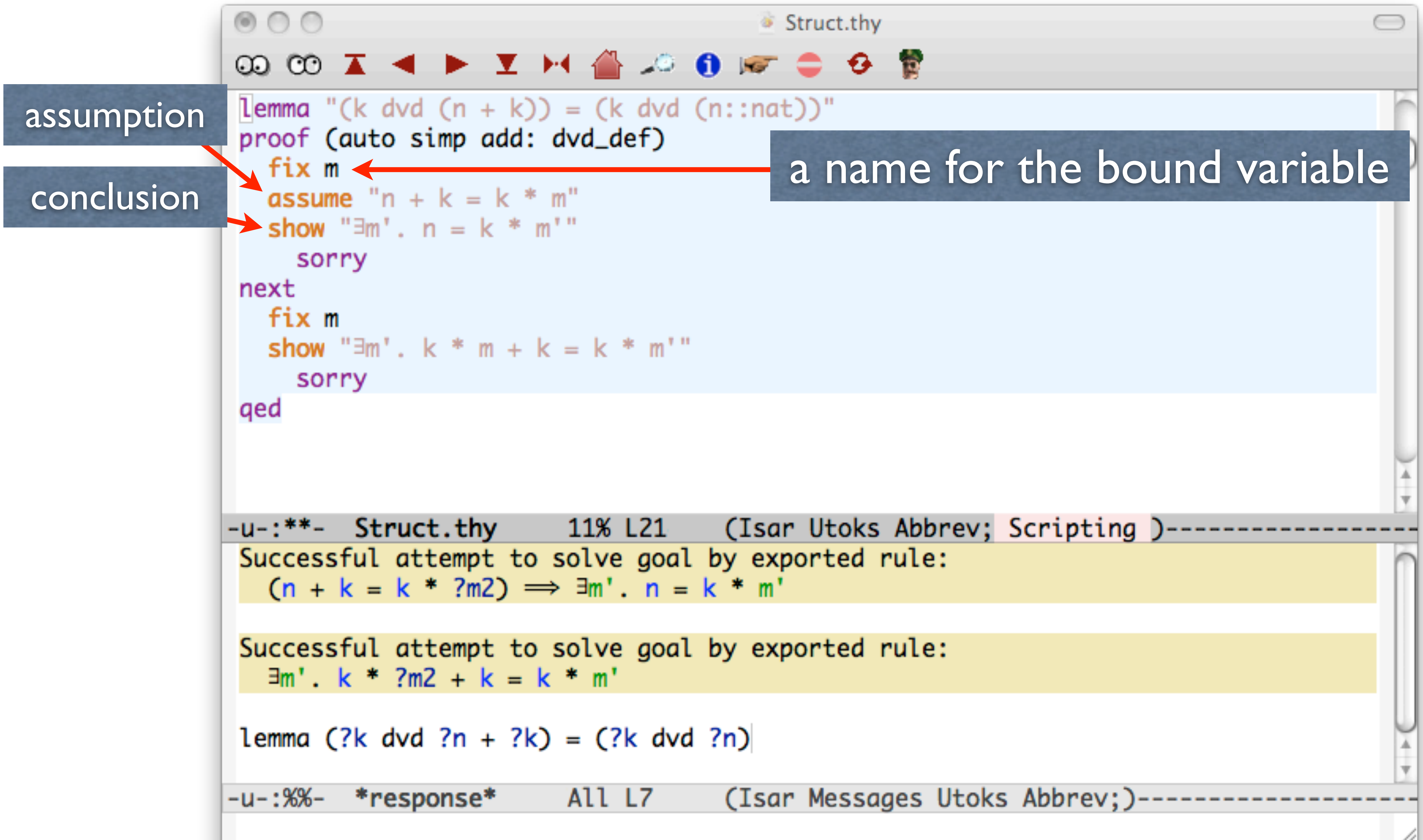

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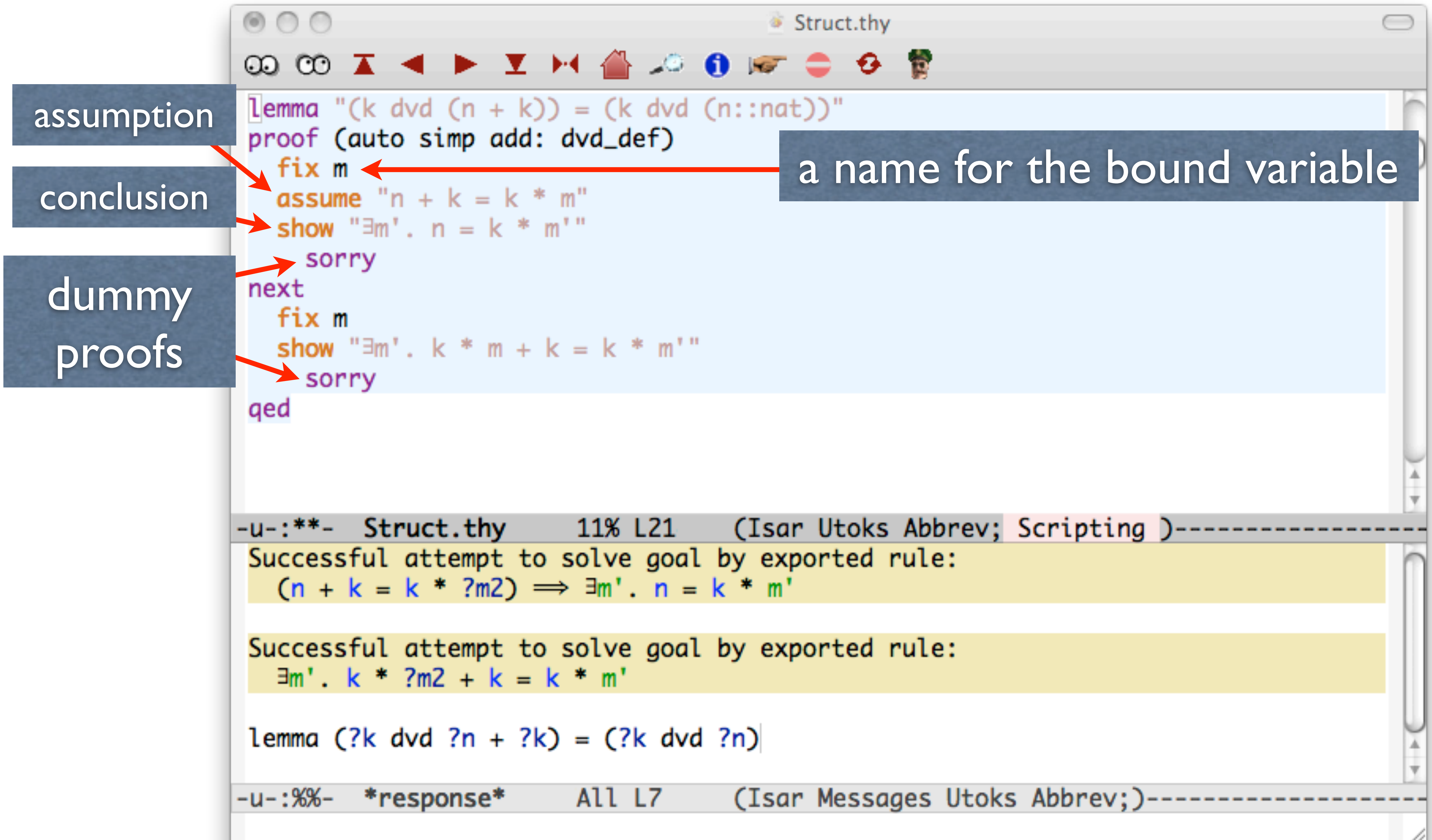

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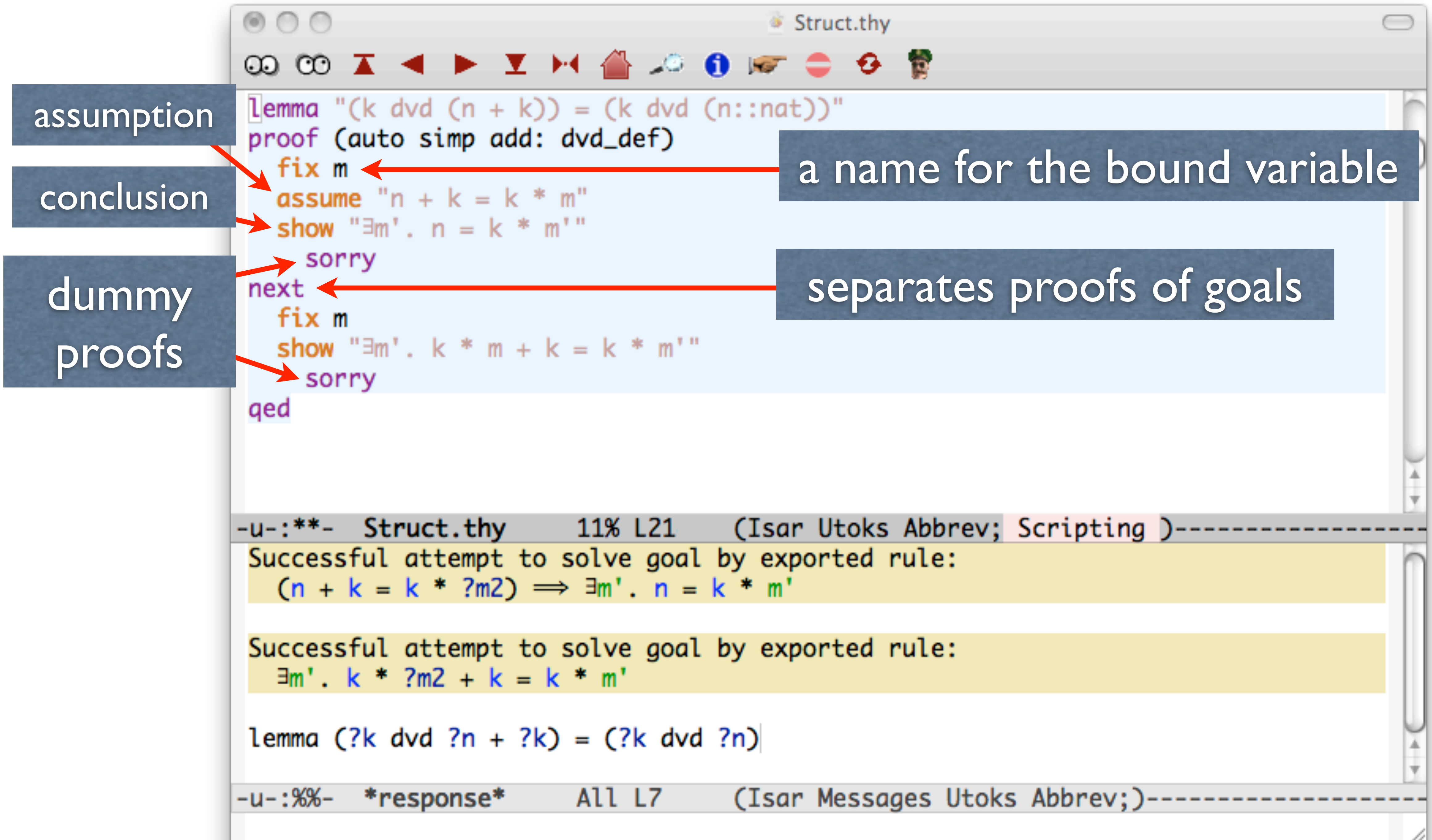

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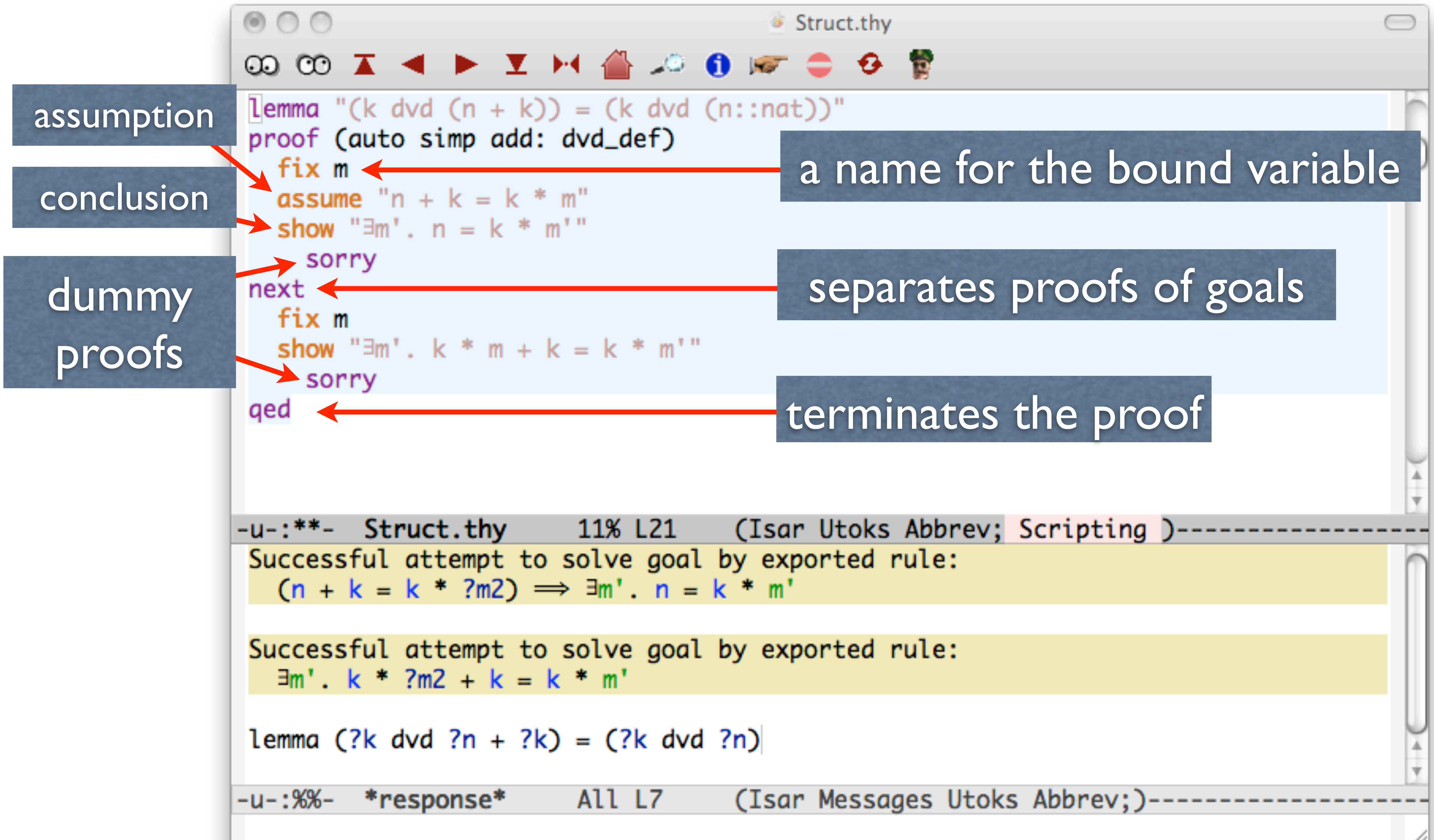
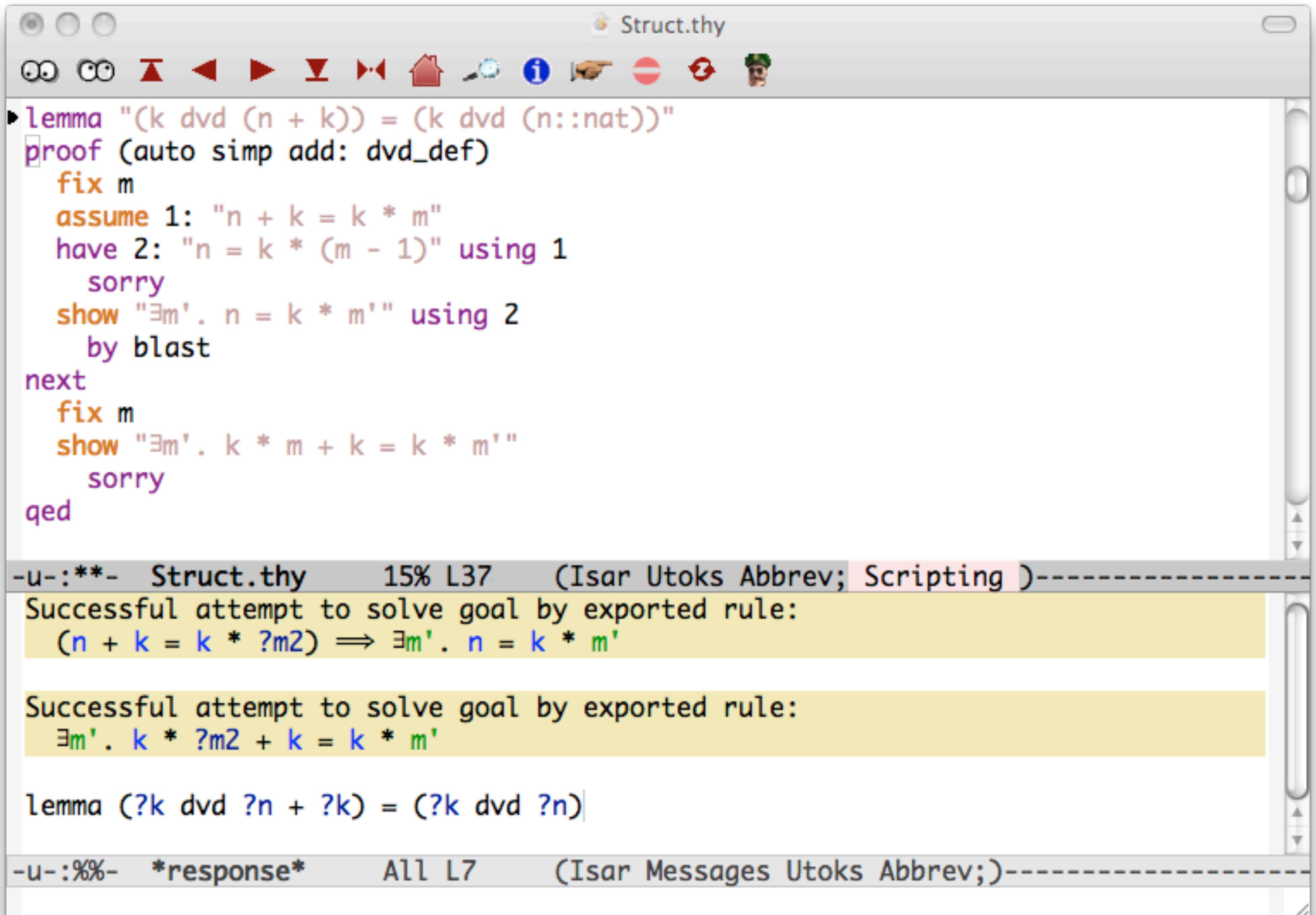


Fleshing Out that Skeleton



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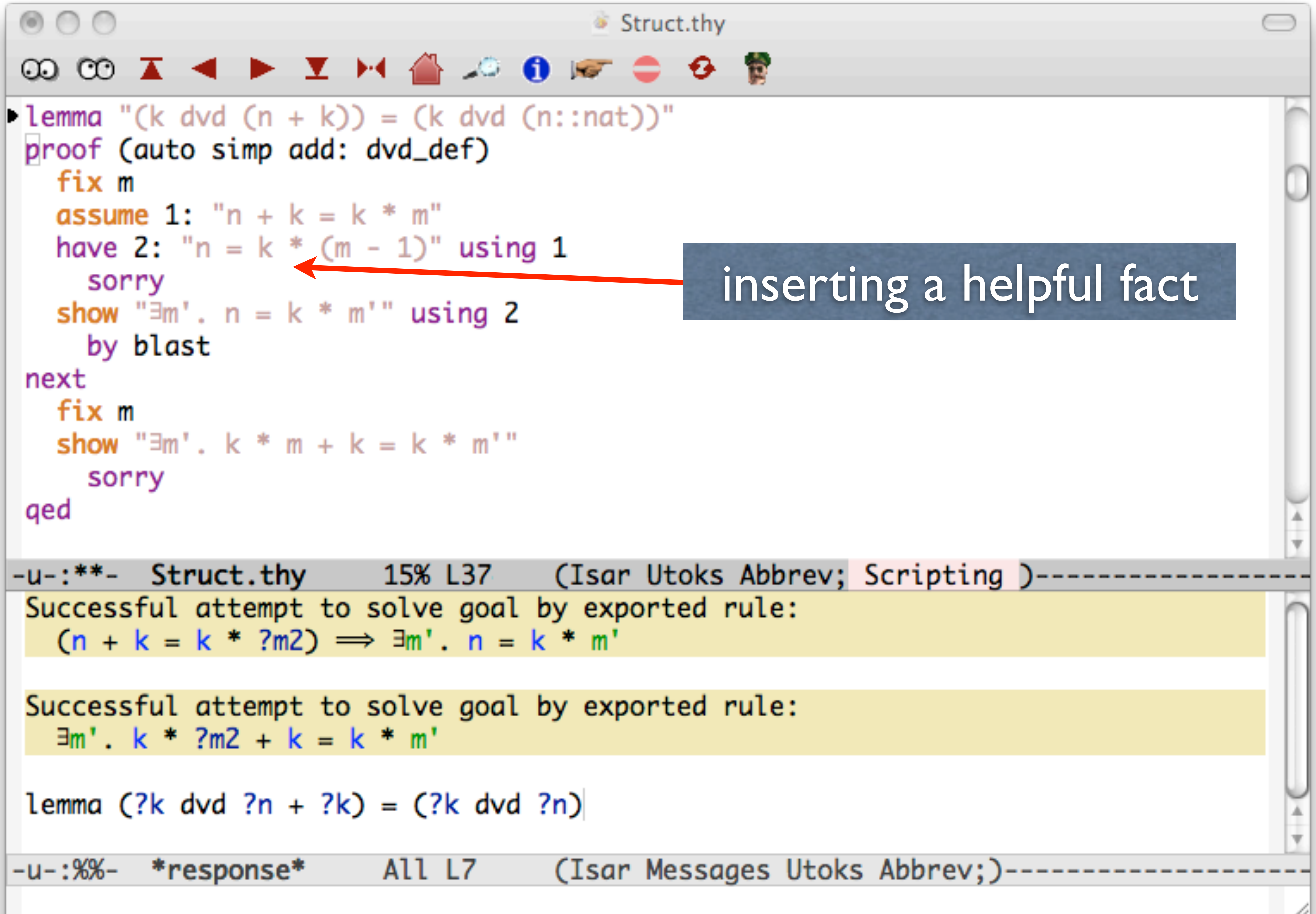

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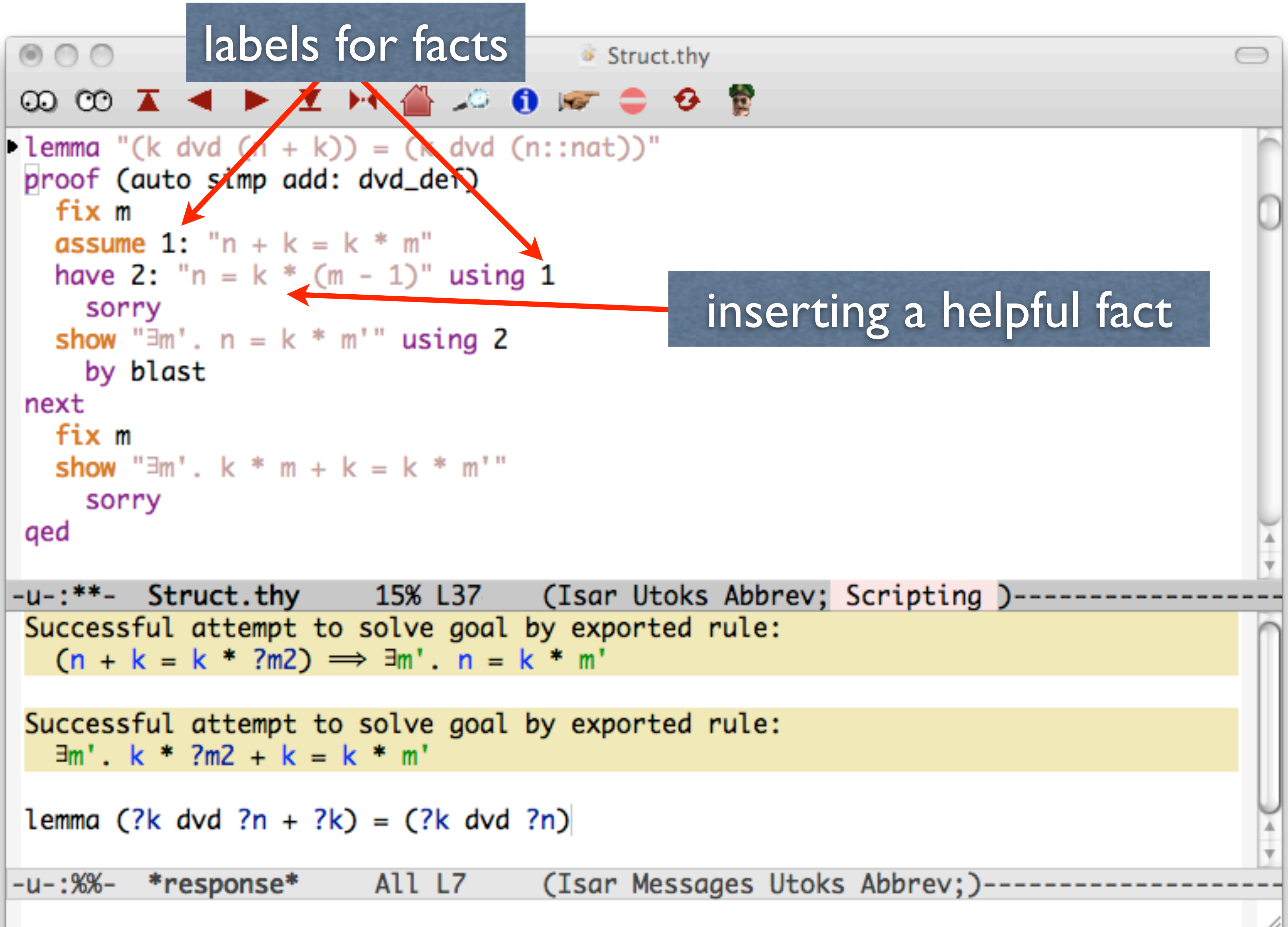

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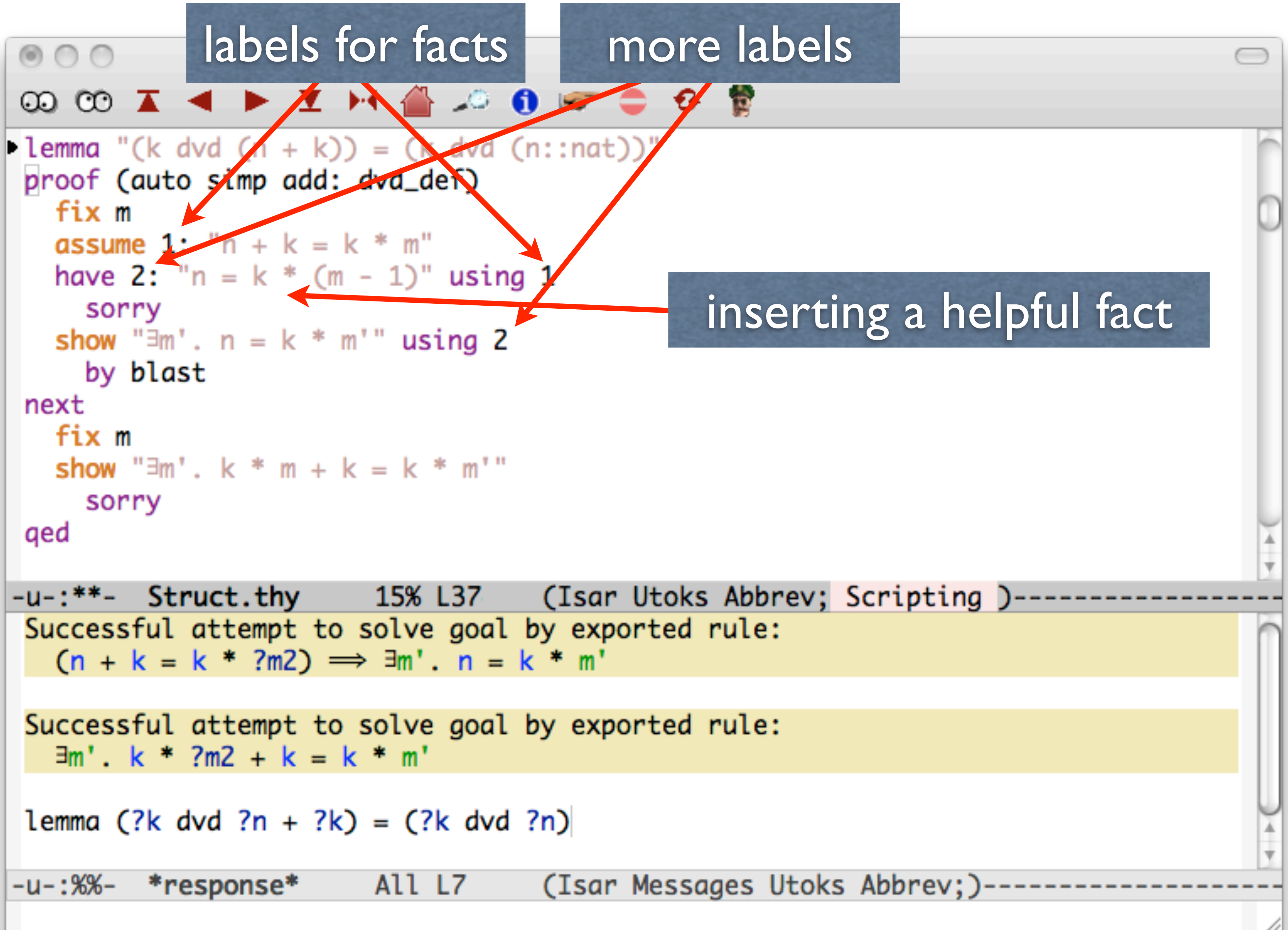

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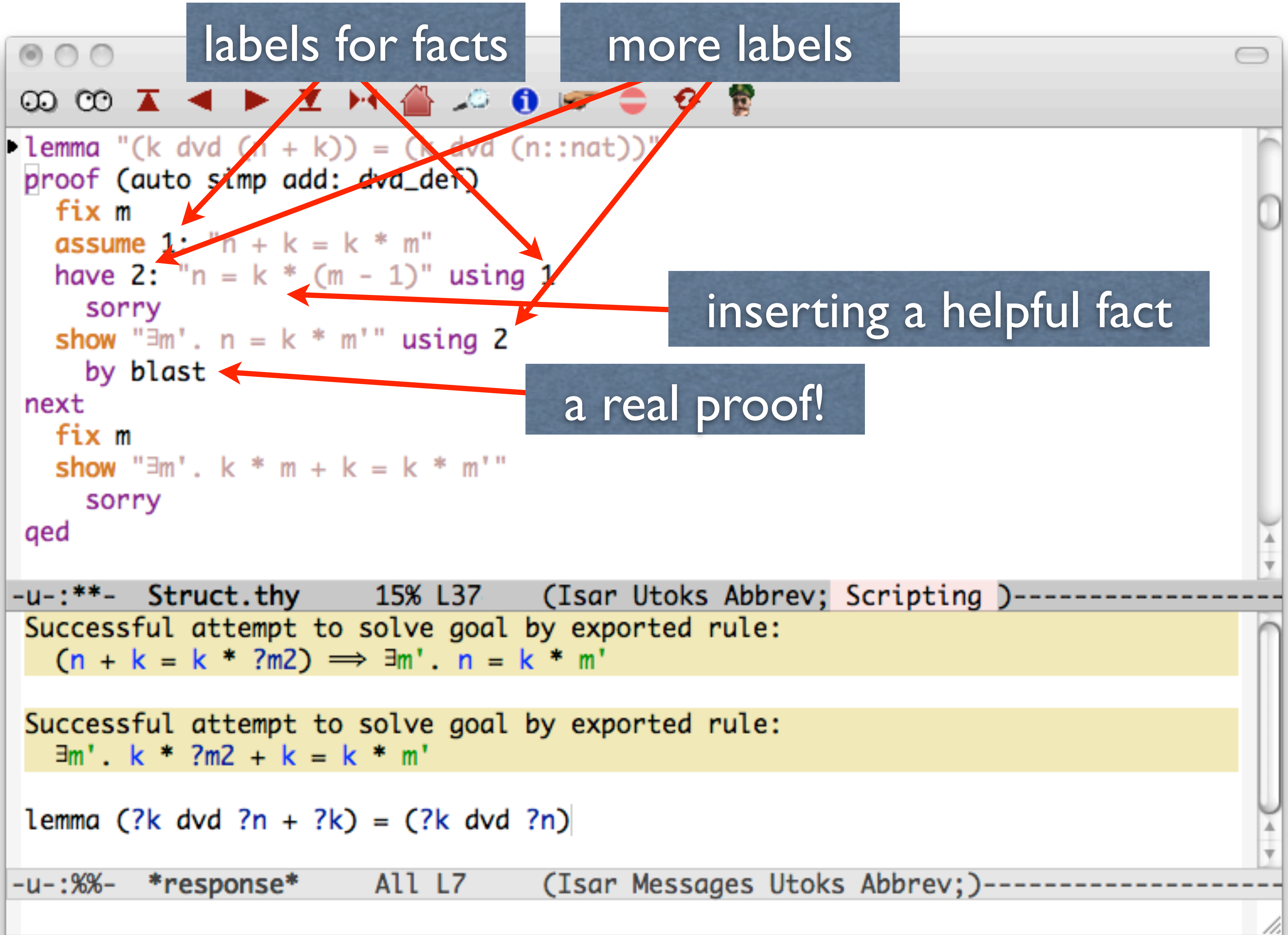
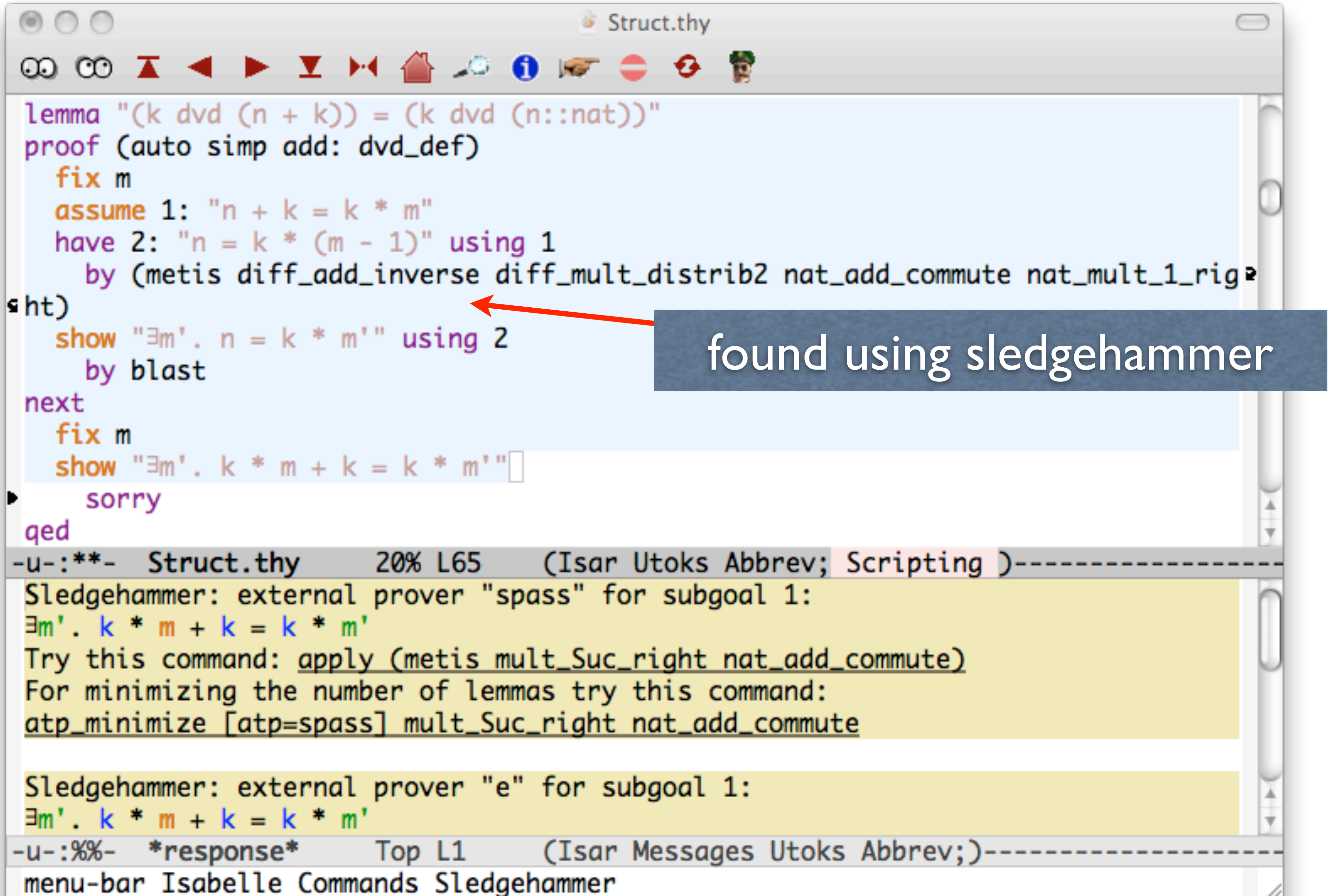


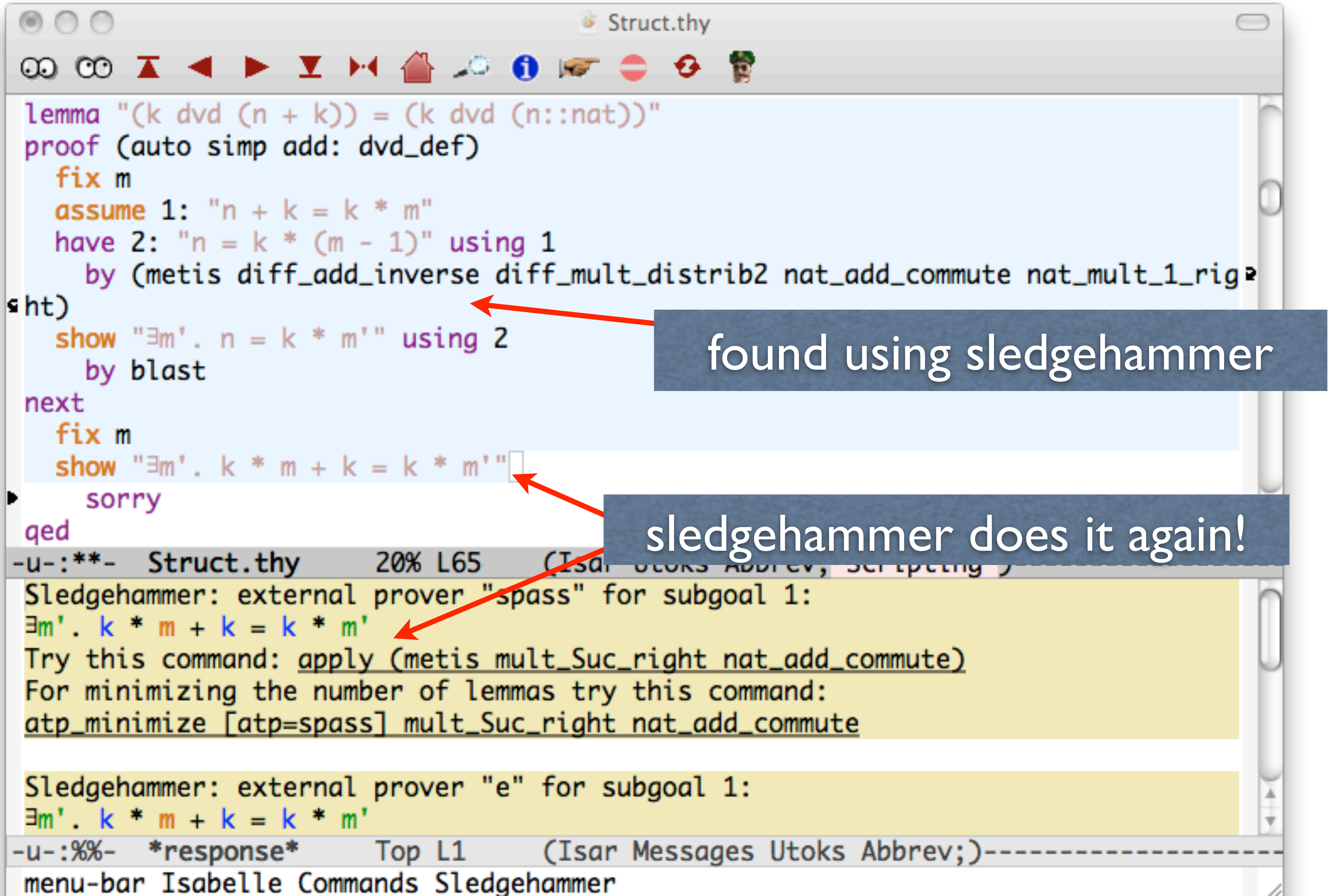
Completing the Proof

```
Struct.thy

lemma "(k dvd (n + k)) = (k dvd (n::nat))"
proof (auto simp add: dvd_def)
  fix m
  assume 1: "n + k = k * m"
  have 2: "n = k * (m - 1)" using 1
    by (metis diff_add_inverse diff_mult_distrib2 nat_add_commute nat_mult_1_rig
ht)
  show "∃m'. n = k * m'" using 2
    by blast
next
  fix m
  show "∃m'. k * m + k = k * m'"
    sorry
qed
-u-:**-  Struct.thy     20% L65     (Isar Utoks Abbrev; Scripting )-----------------
Sledgehammer: external prover "spass" for subgoal 1:
∃m'. k * m + k = k * m'
Try this command: apply (metis mult_Suc_right nat_add_commute)
For minimizing the number of lemmas try this command:
atp_minimize [atp=spass] mult_Suc_right nat_add_commute

Sledgehammer: external prover "e" for subgoal 1:
∃m'. k * m + k = k * m'
-u-:%%-  *response*     Top L1      (Isar Messages Utoks Abbrev;)-------------------
menu-bar Isabelle Commands Sledgehammer
```

Completing the Proof



Completing the Proof



Streamlining the Proof

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Streamlining the Proof

```
assume 1: "n + k = k * m"                     →  assume "n + k = k * m"
have 2: "n = k * (m - 1)" using 1             →  hence "n = k * (m - 1)"
  by (metis diff_add_inverse diff                  by (metis diff_add_inverse diff
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```
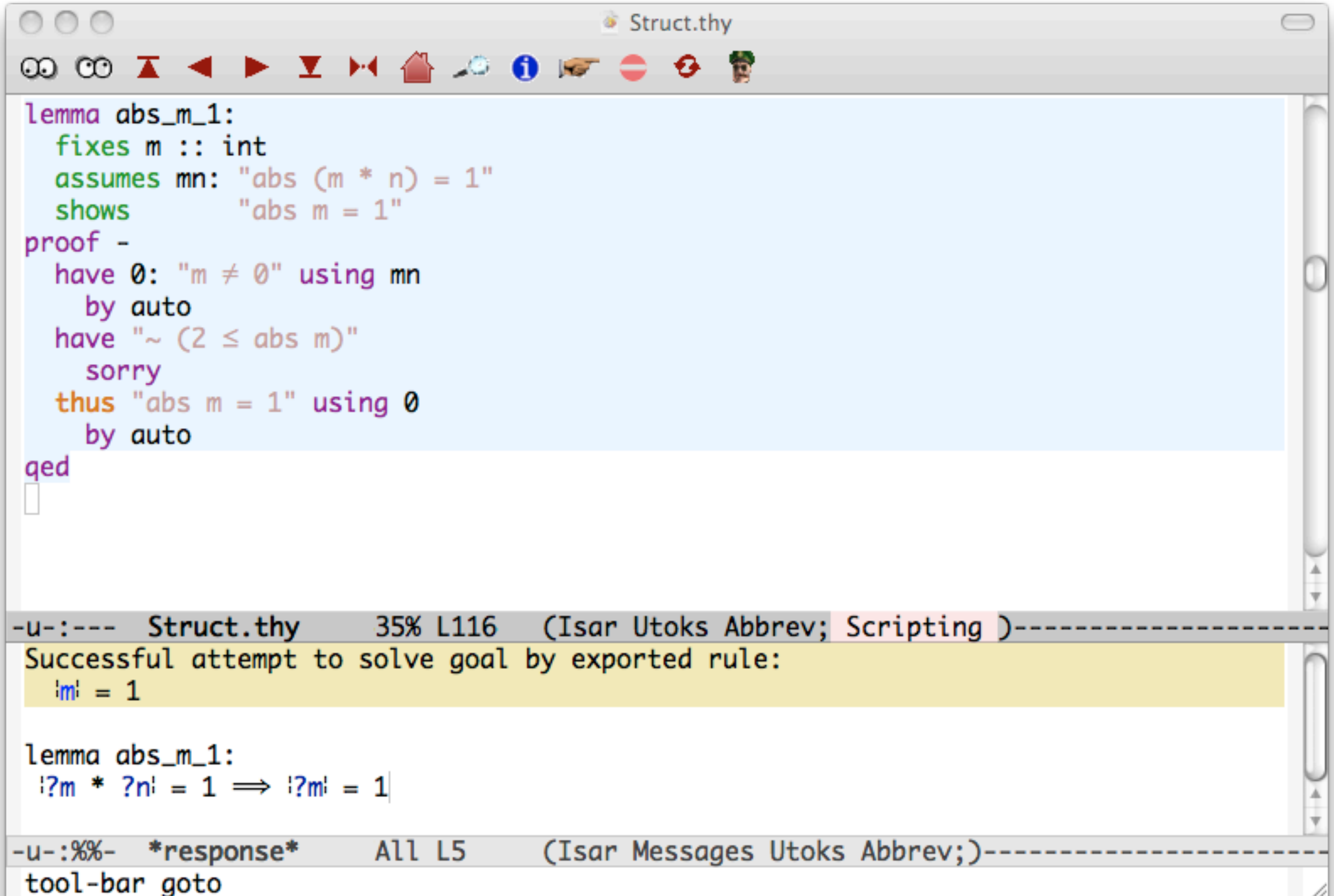

- `hence` means `have` — `using` the previous fact
- `thus` means `show` — `using` the previous fact
- There are numerous other tricks of this sort!

Another Proof Skeleton



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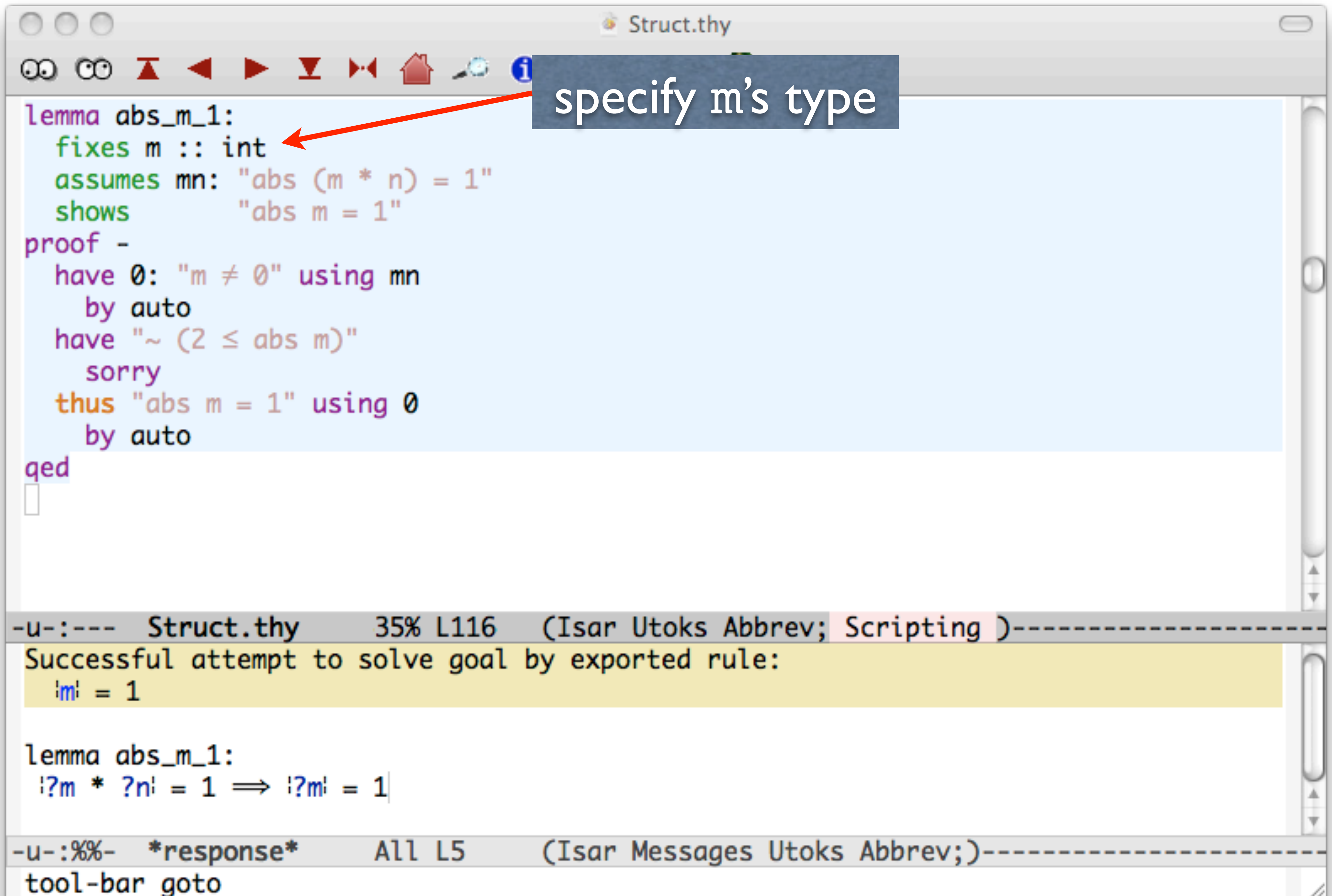

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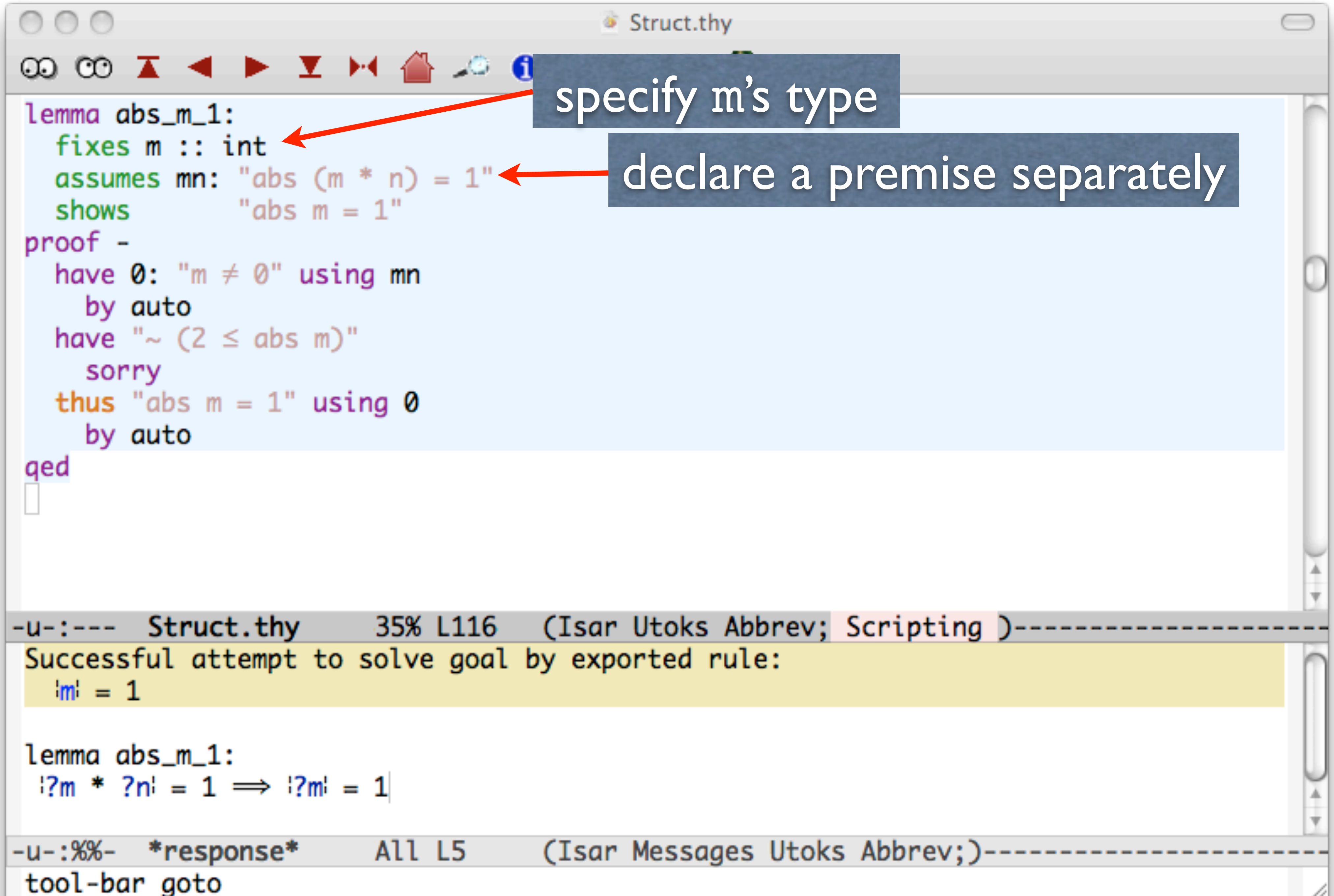

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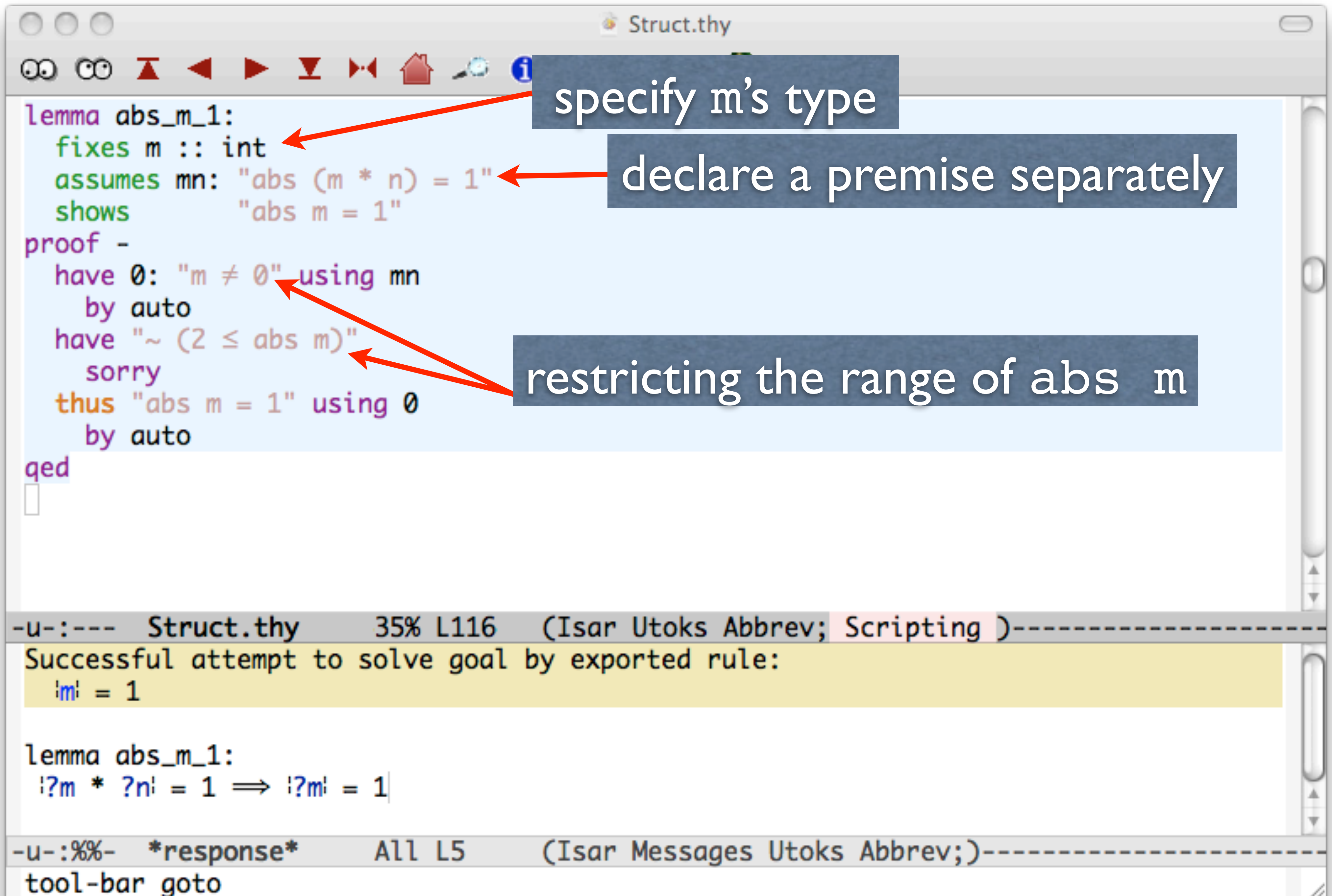

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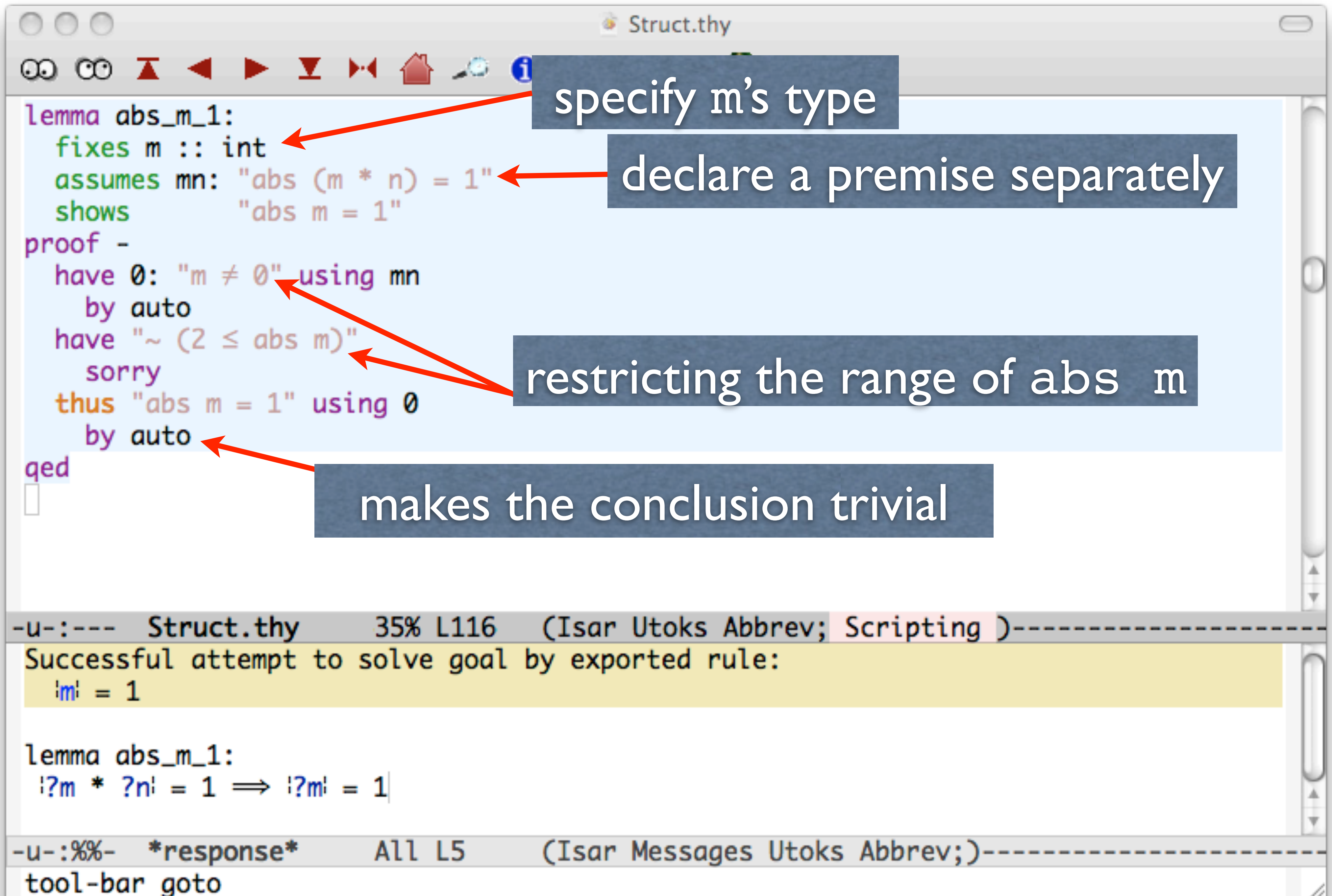

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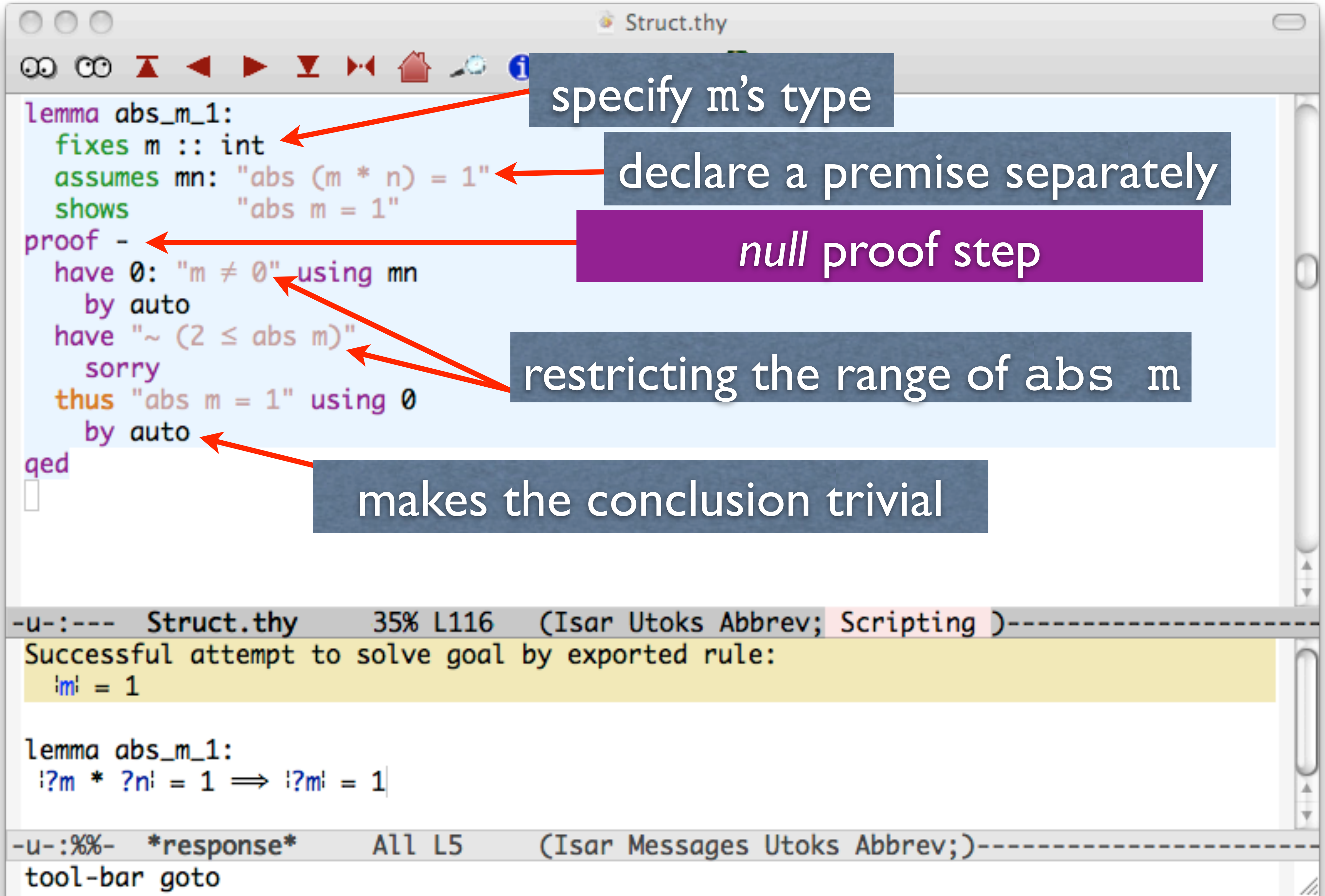
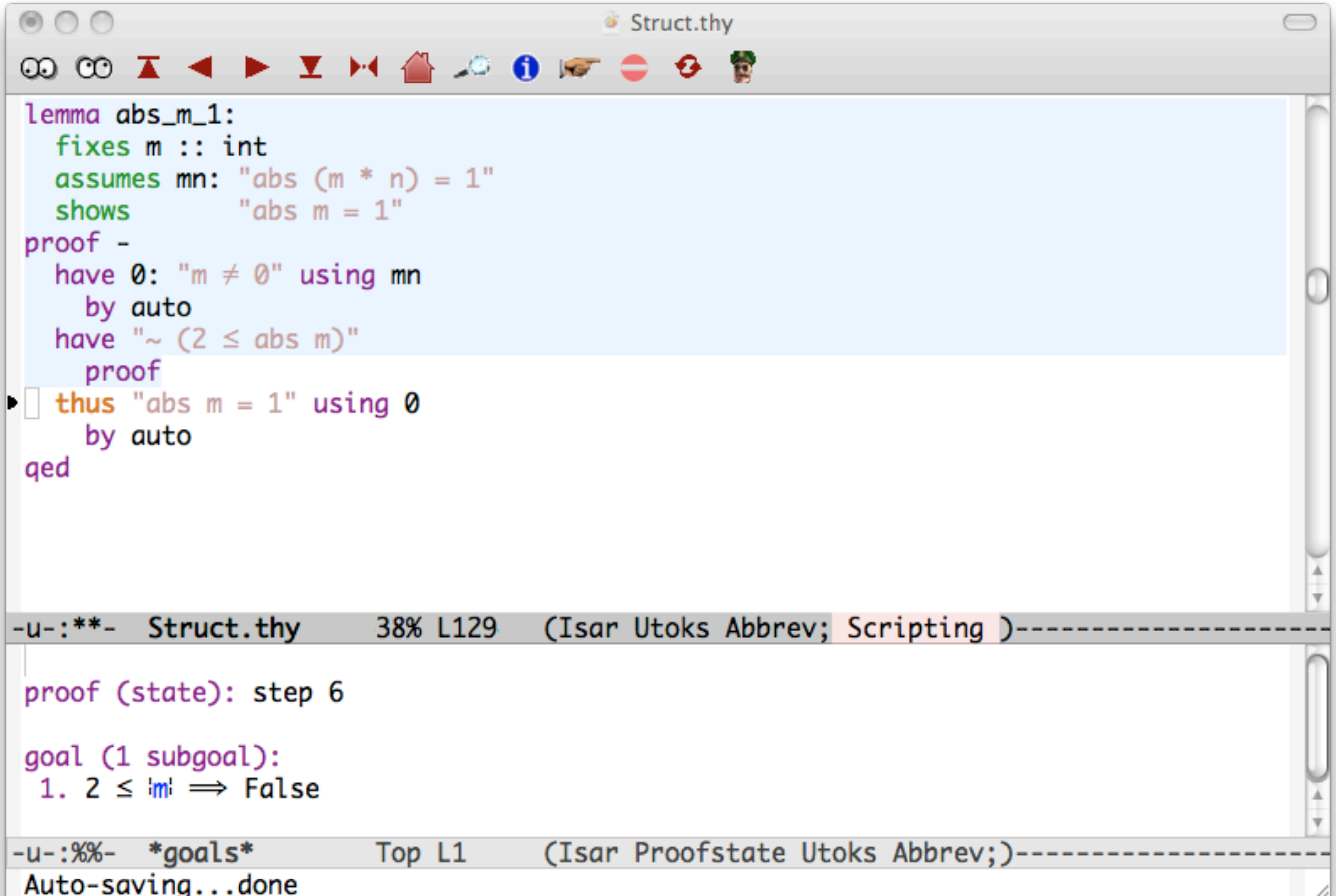


Starting a Nested Proof



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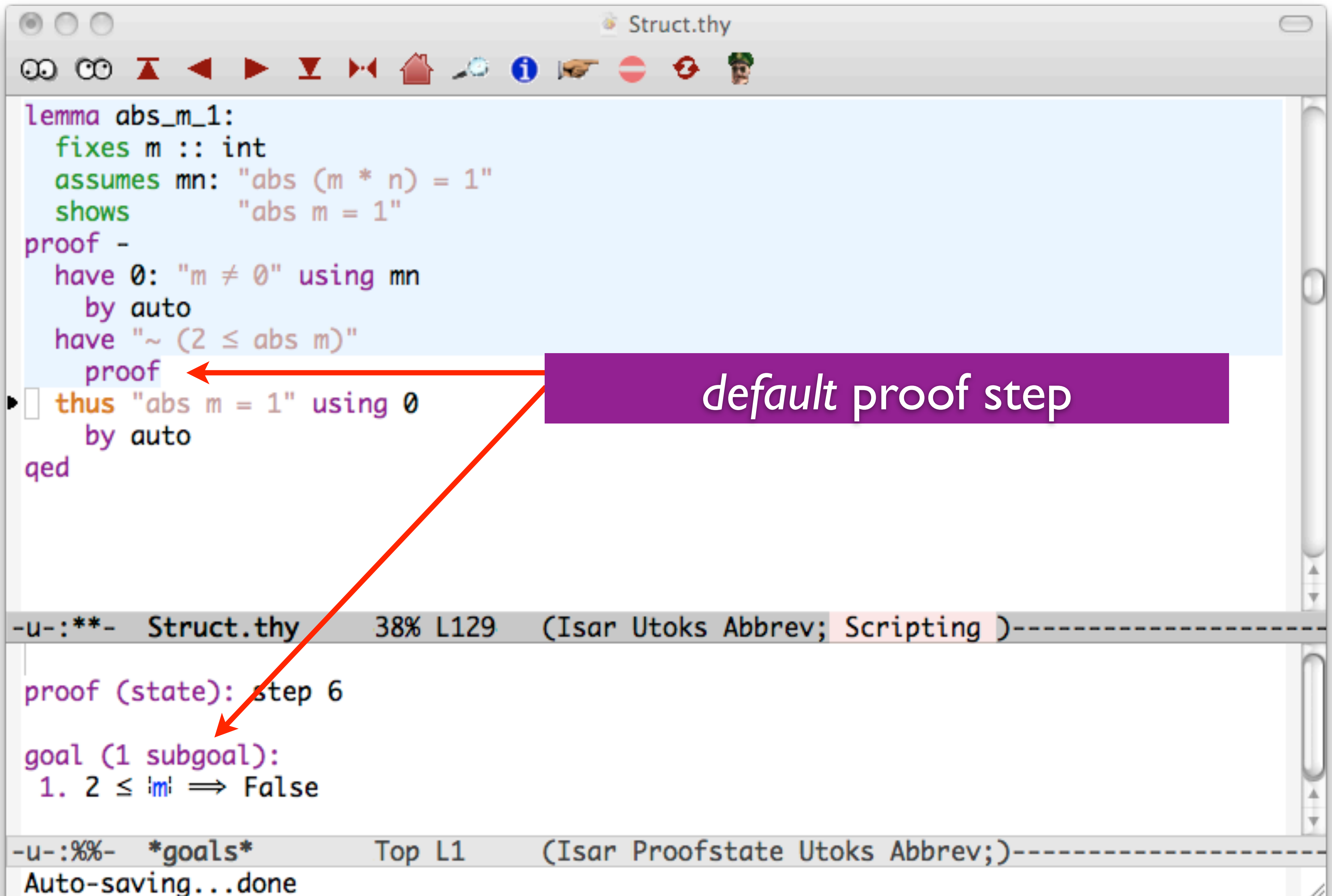

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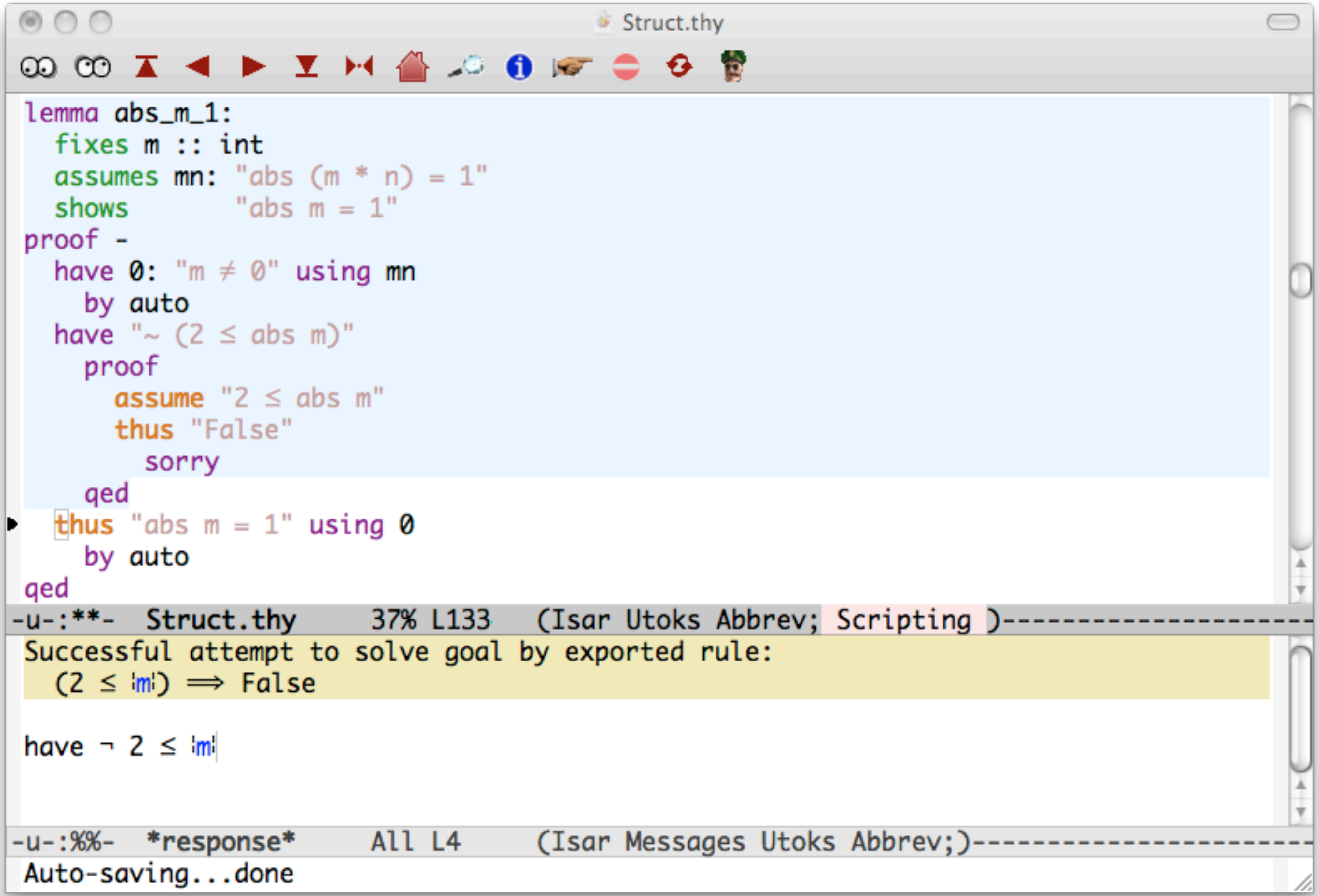

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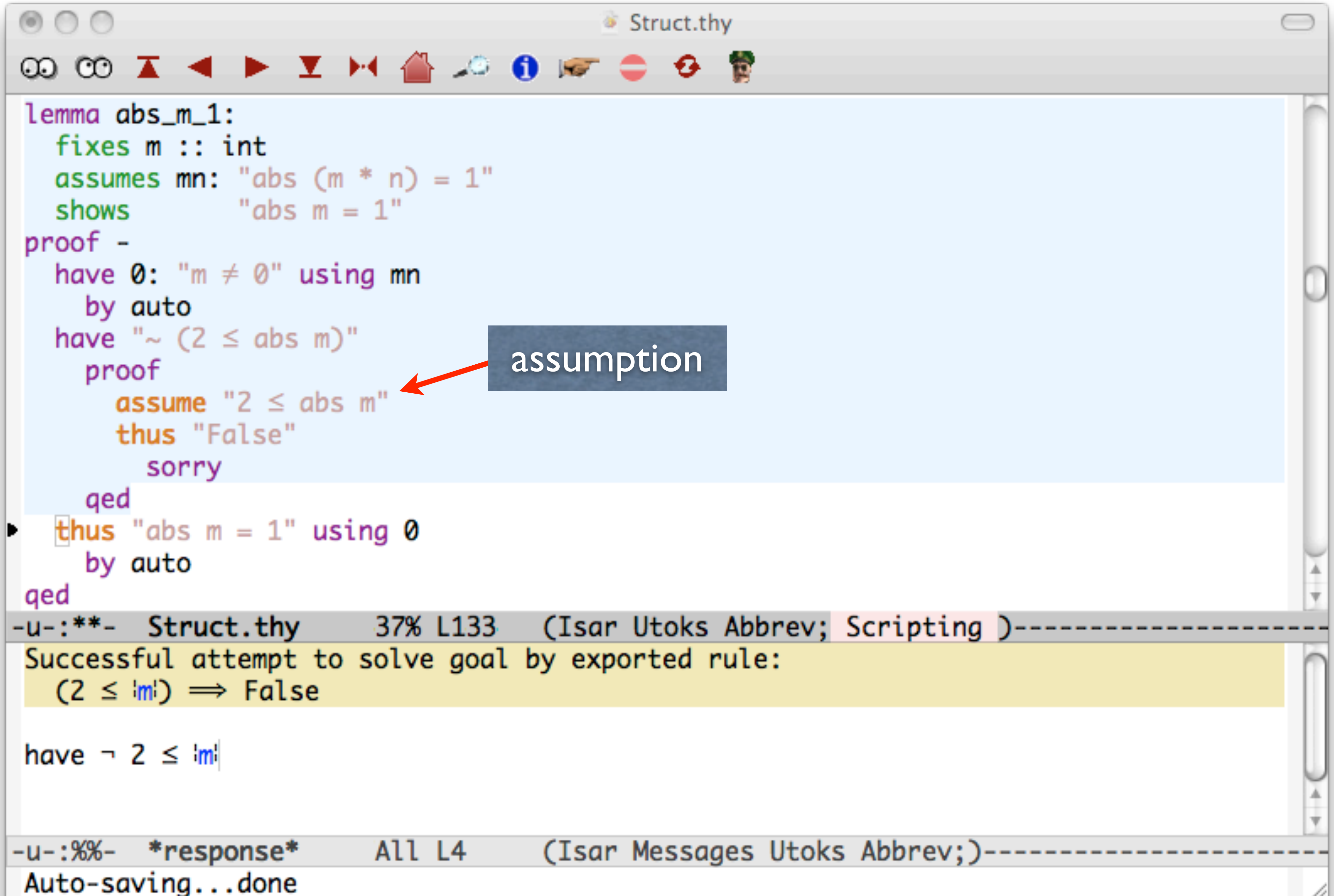

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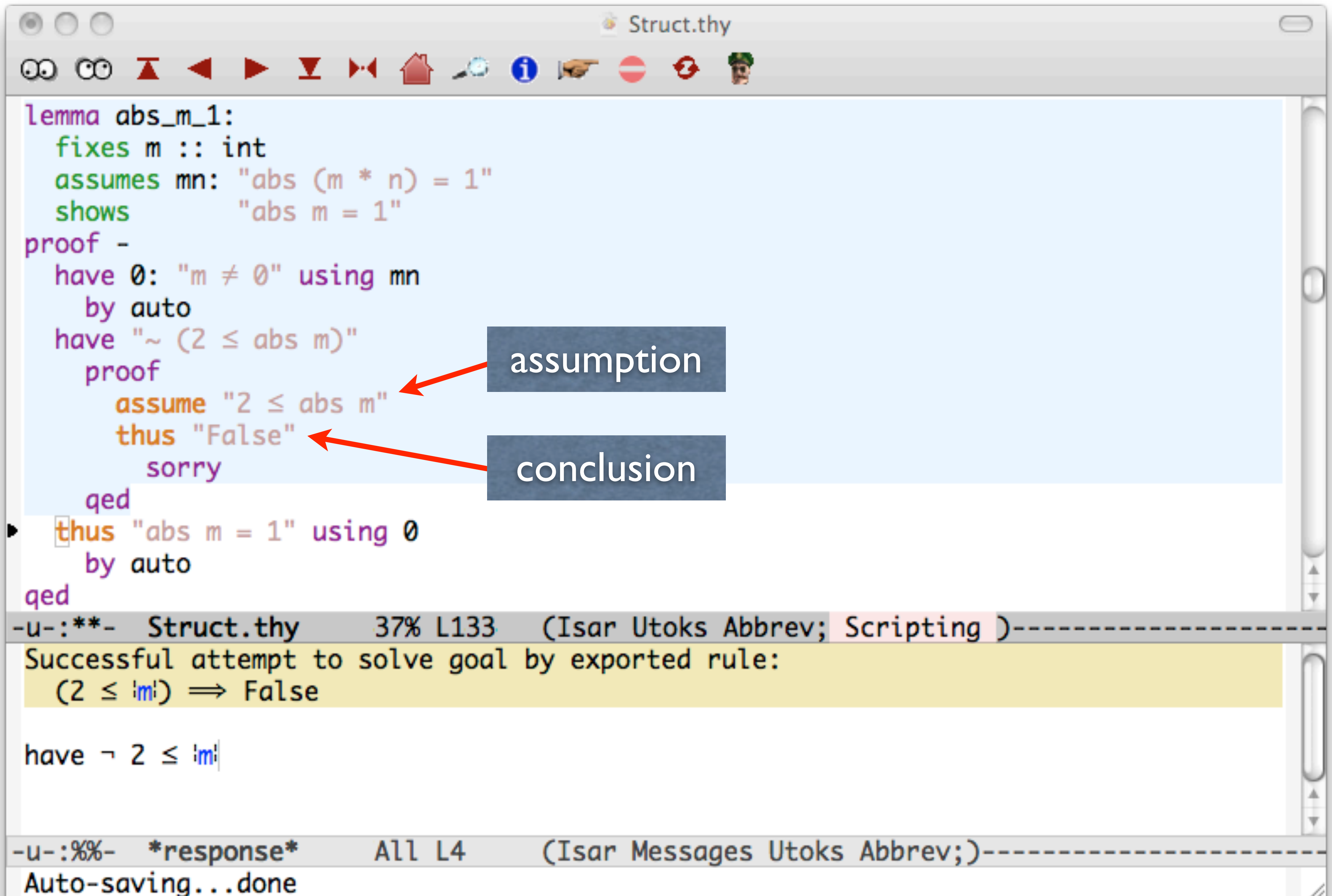
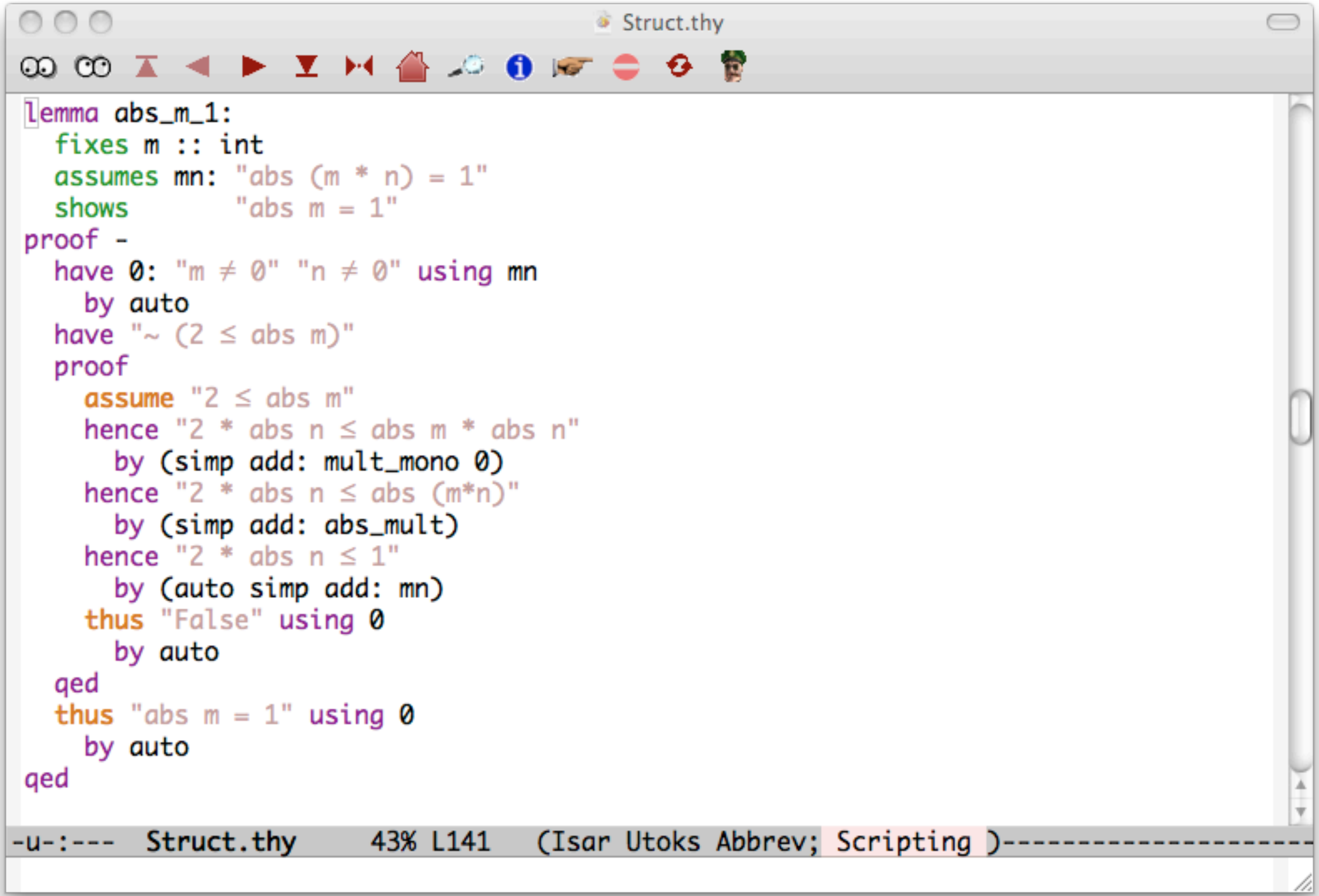


A Complete Proof



A Complete Proof

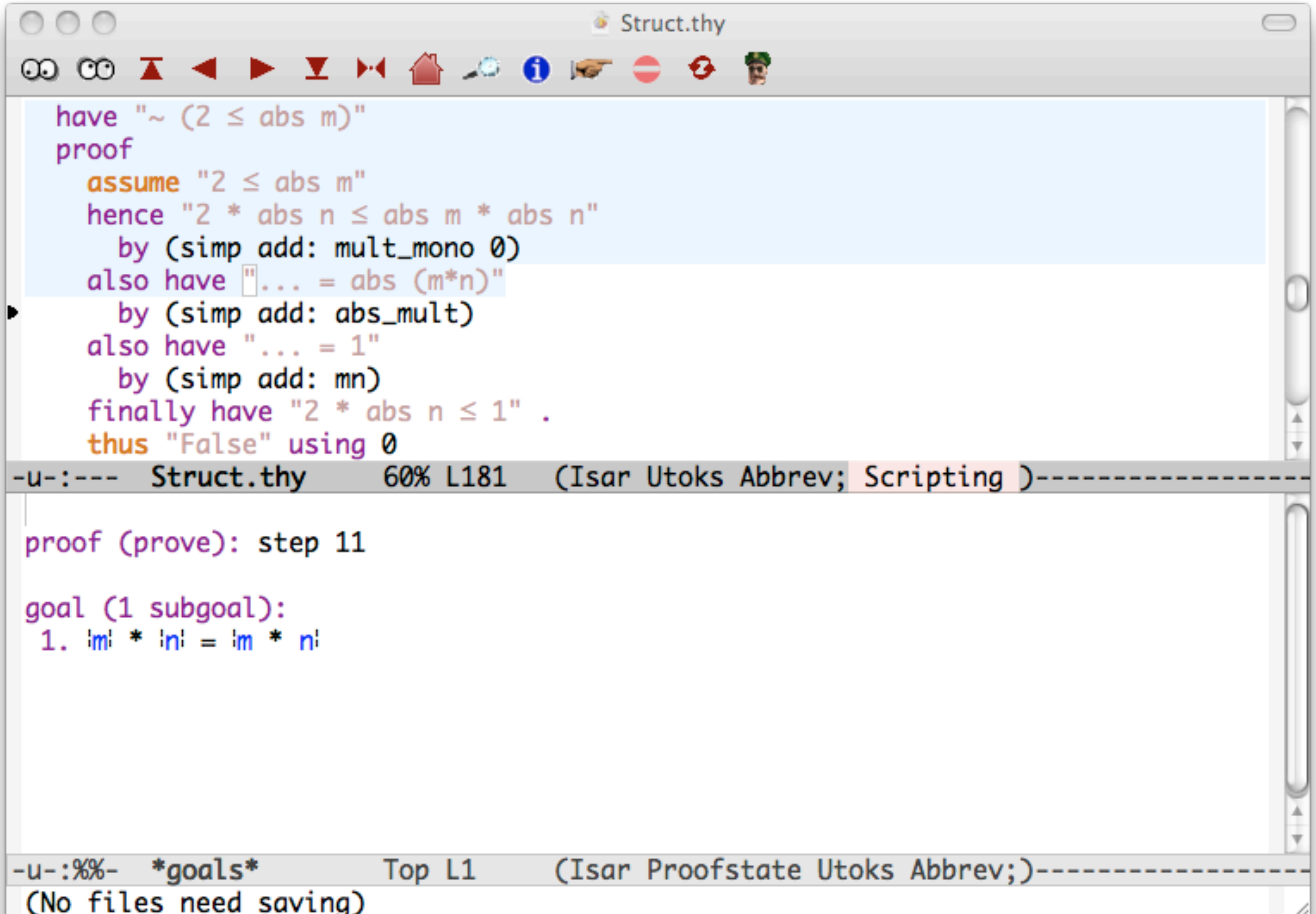
Struct.thy

```
lemma abs_m_1:
  fixes m :: int
  assumes mn: "abs (m * n) = 1"
  shows       "abs m = 1"
proof -
  have 0: "m ≠ 0" "n ≠ 0" using mn
    by auto
  have "~ (2 ≤ abs m)"
  proof
    assume "2 ≤ abs m"
    hence "2 * abs n ≤ abs m * abs n"
      by (simp add: mult_mono 0)
    hence "2 * abs n ≤ abs (m*n)"
      by (simp add: abs_mult)
    hence "2 * abs n ≤ 1"
      by (auto simp add: mn)
    thus "False" using 0
      by auto
  qed
  thus "abs m = 1" using 0
    by auto
qed
```

-u-:--- Struct.thy 43% L141 (Isar Utoks Abbrev; Scripting)--------------------

a chain of steps leads to contradiction

Calculational Proofs



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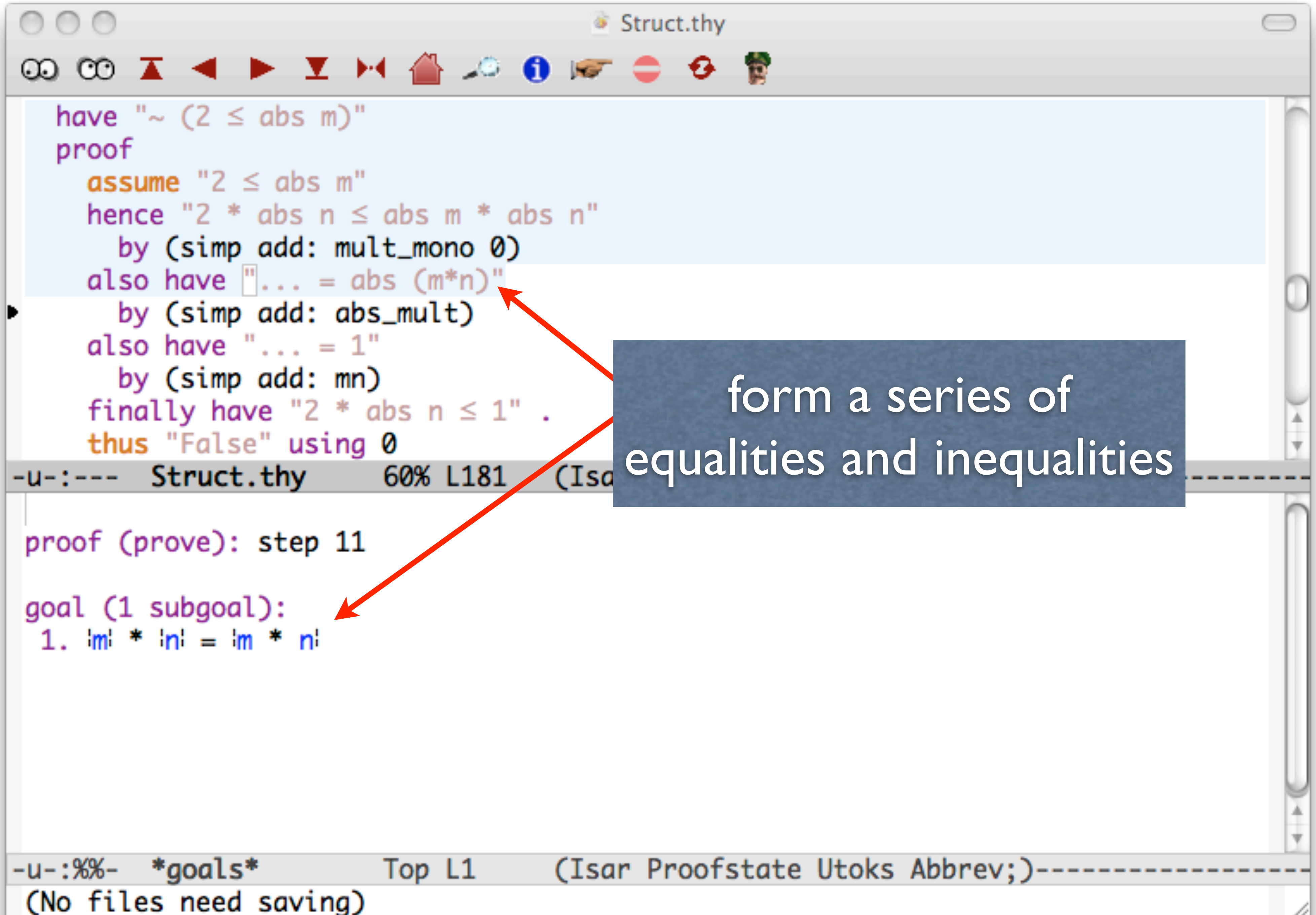
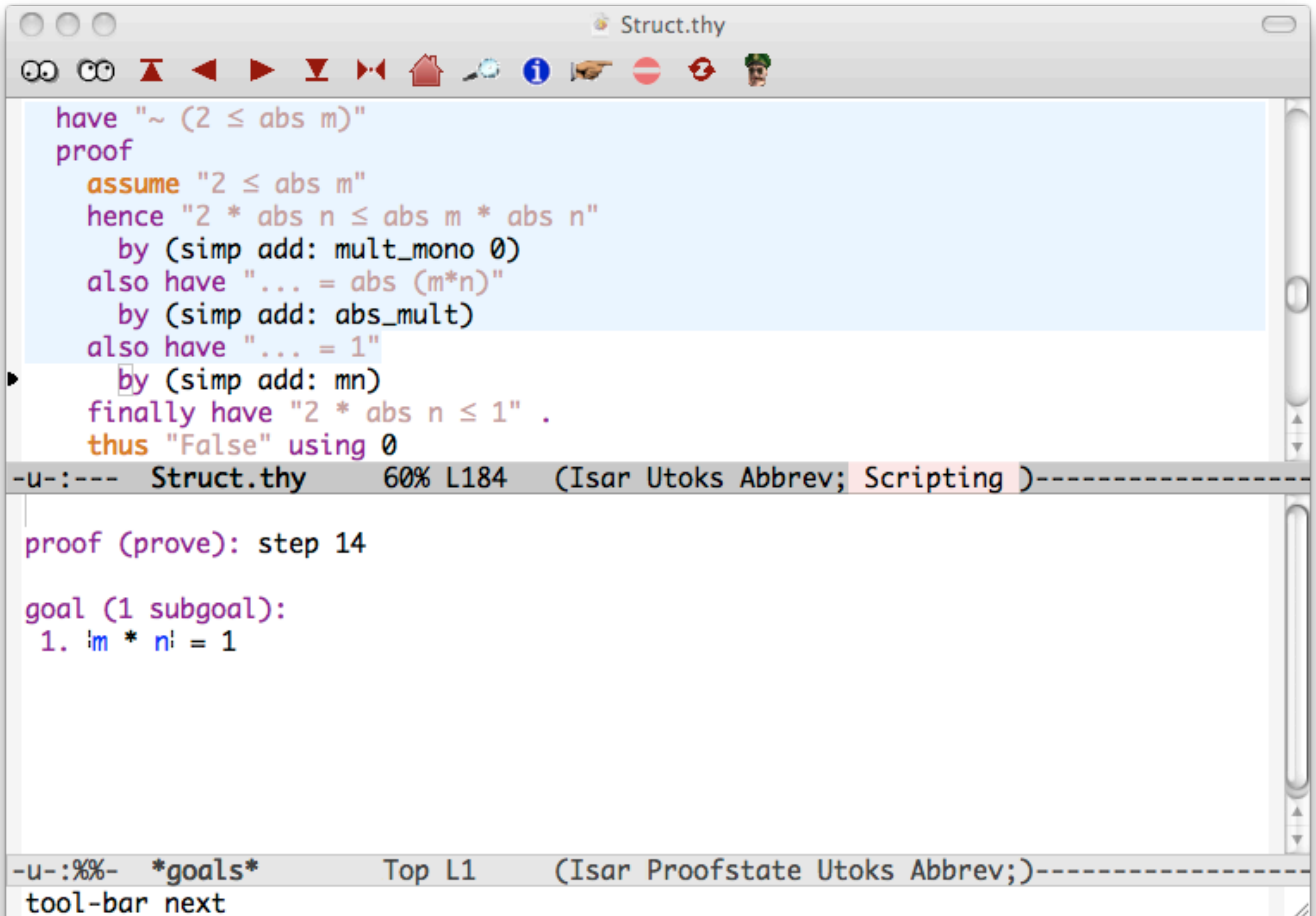


The Next Step



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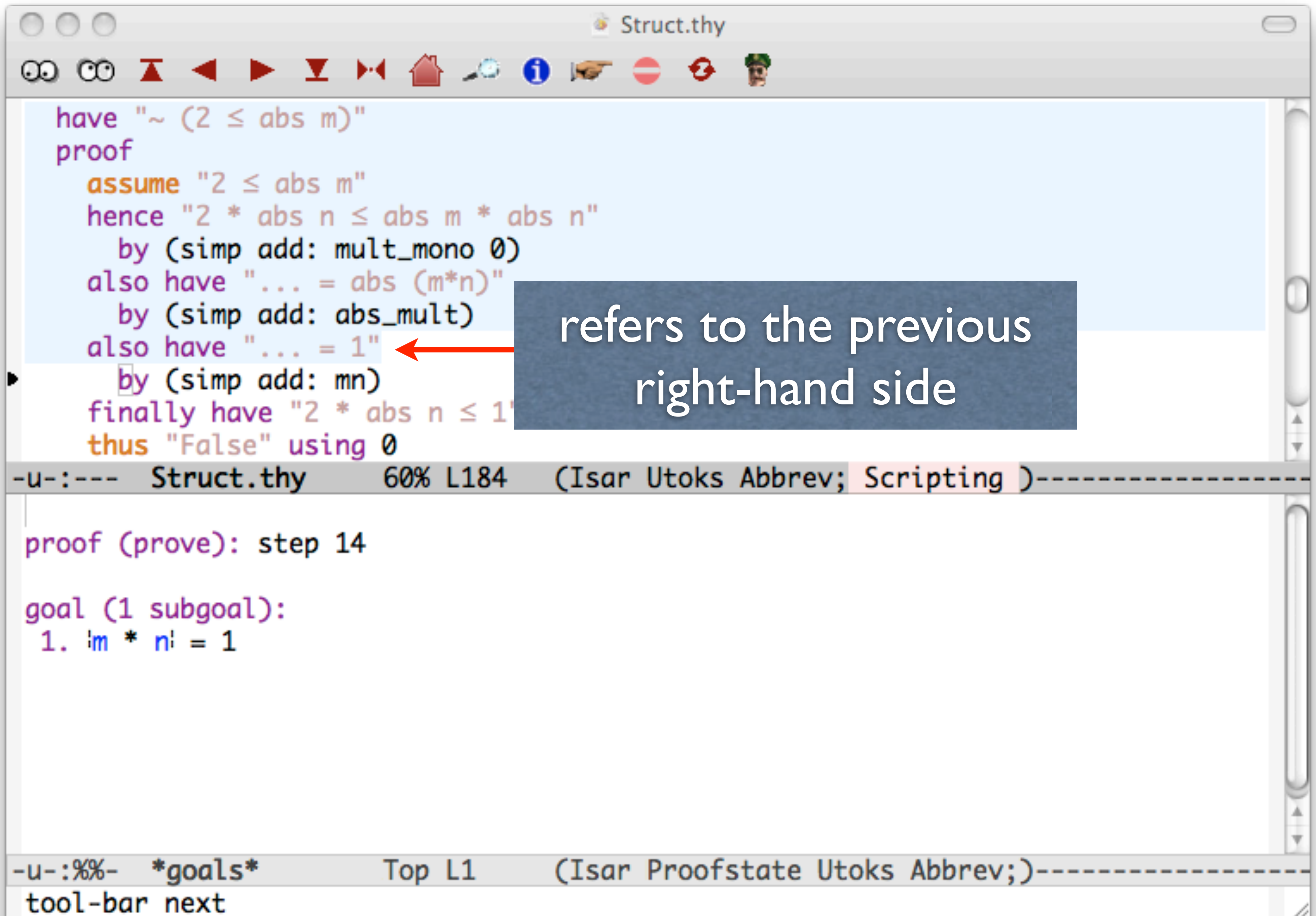
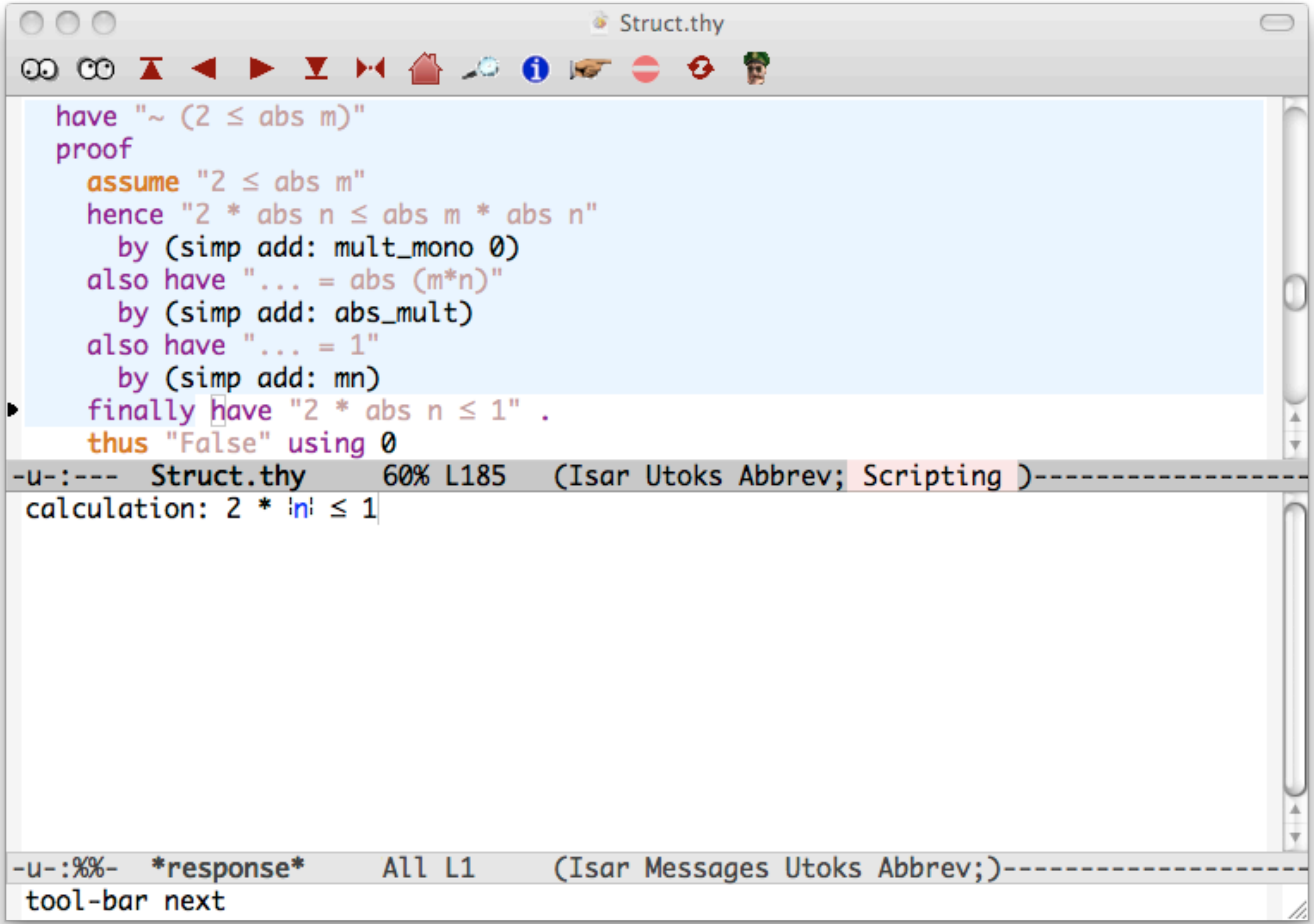


The Internal Calculation



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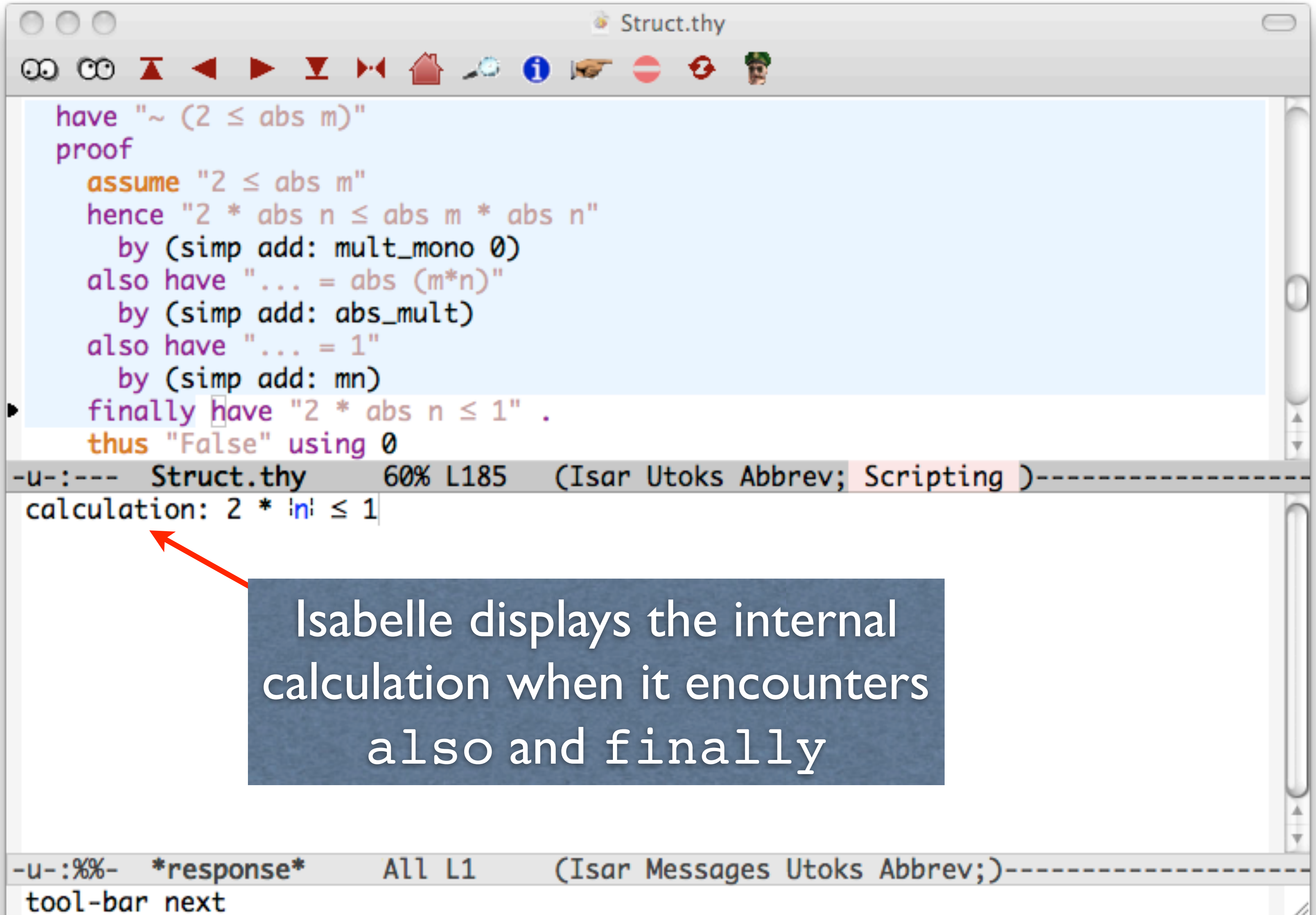

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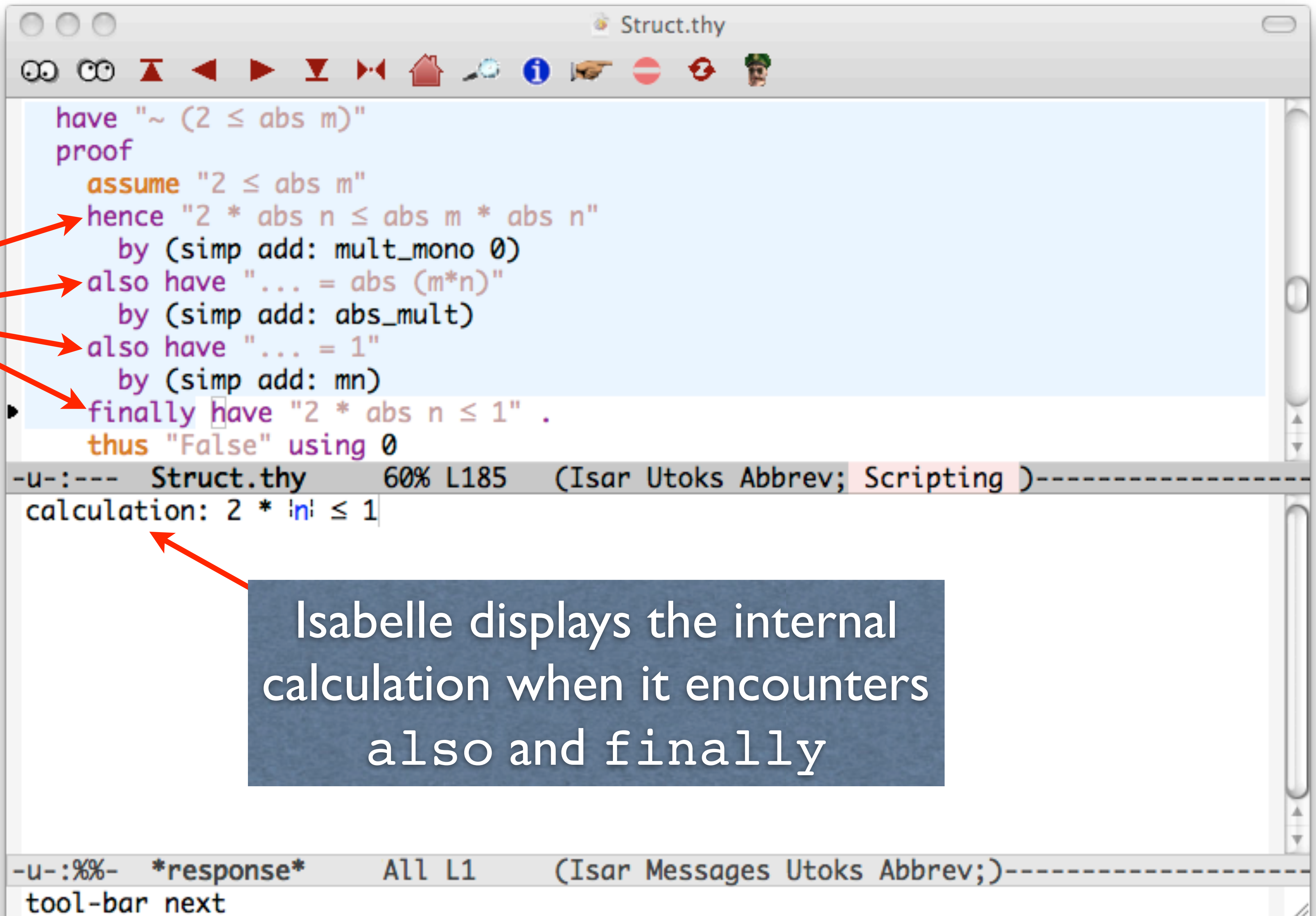
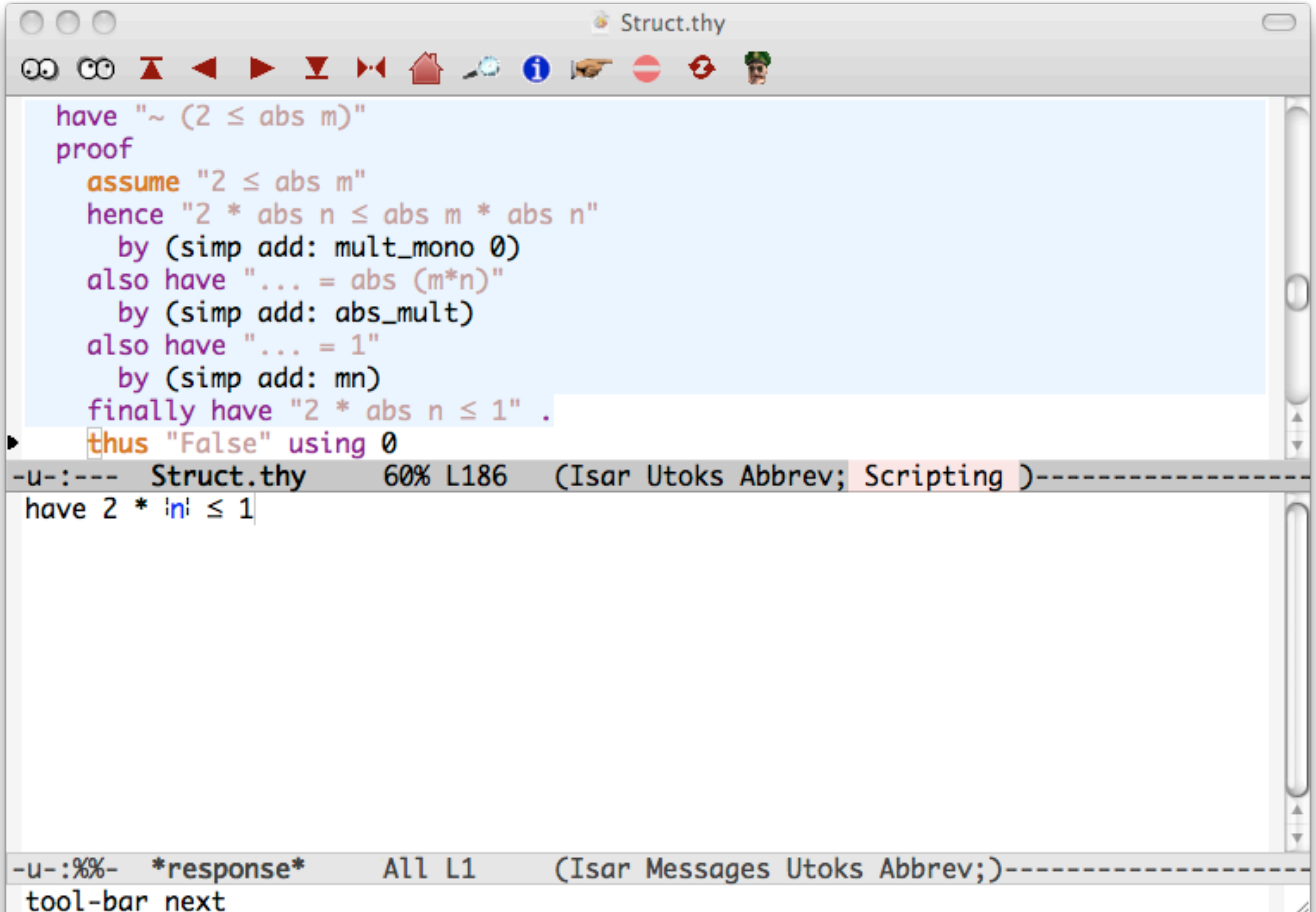


structure of a calculation

Isabelle displays the internal calculation when it encounters `also` and `finally`

Ending the Calculation



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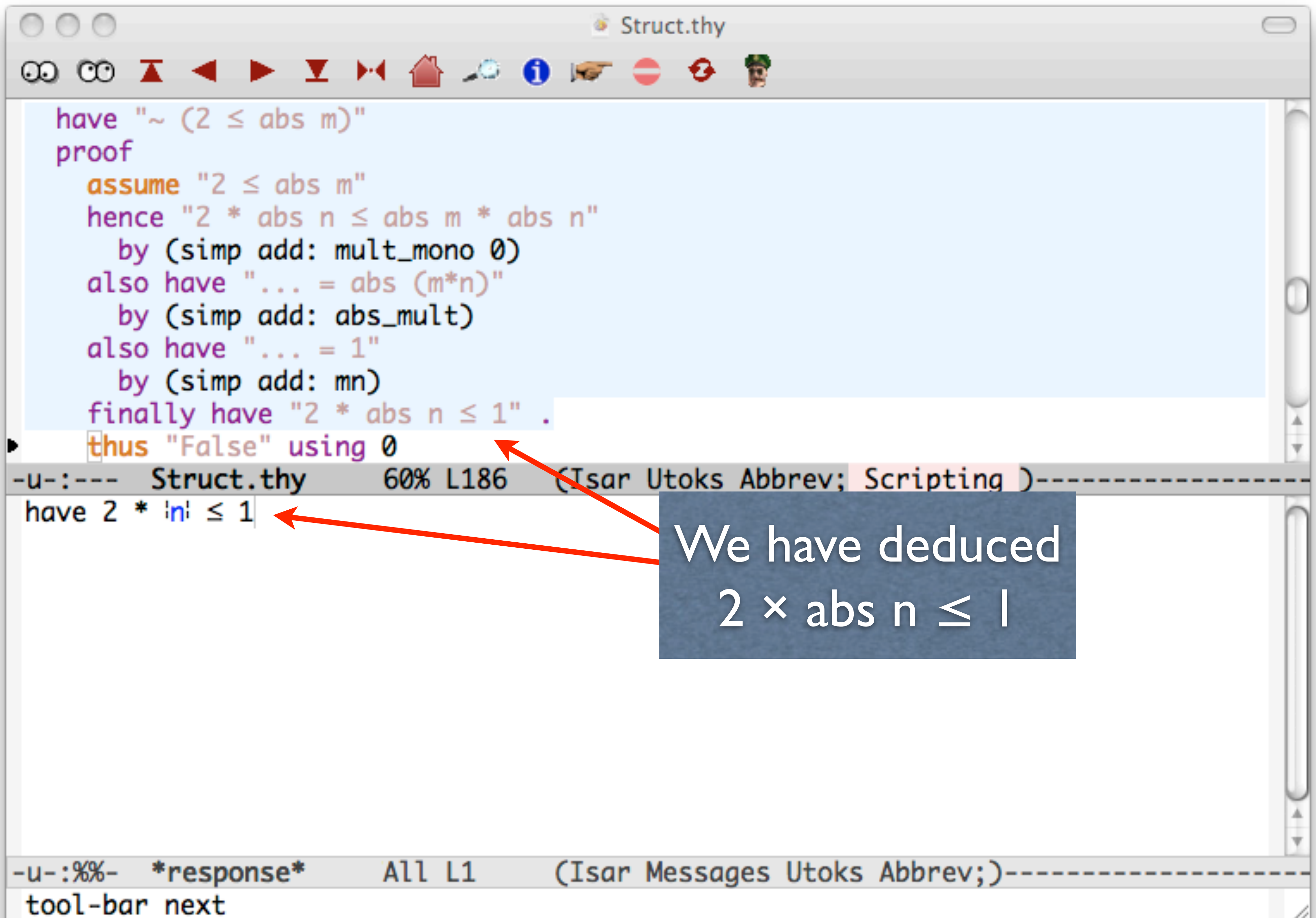

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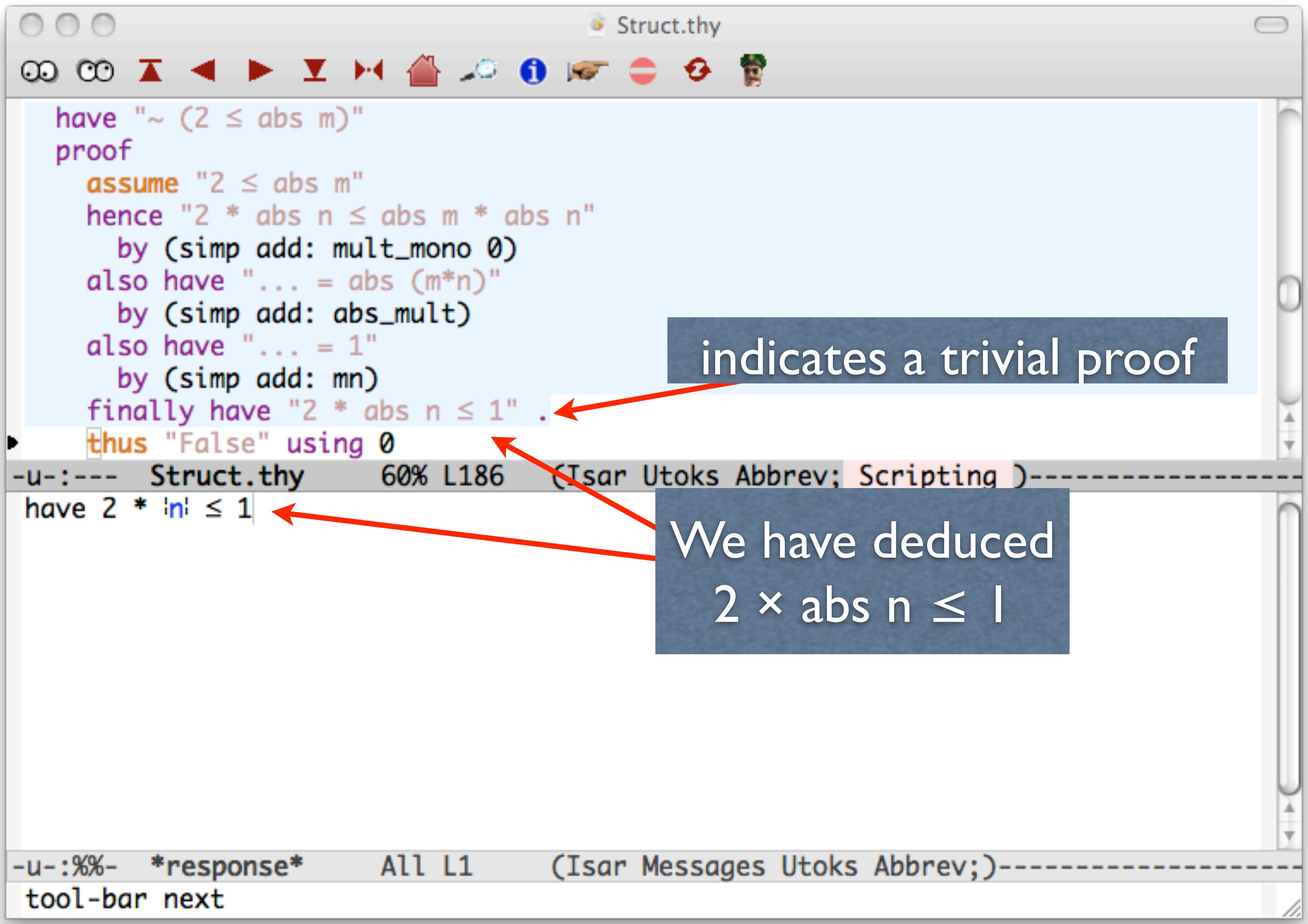

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- Subsequent lines begin, `also have “... = “`
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- The concluding line begins, `finally have/show`, repeats the calculation and terminates with `(.)`